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Williamson Type Matrices through Pairwise Balanced Design

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Abstract

In this paper it is shown that a new type of non-circulant symmetric Williamson matrices can be constructed through a new family of Pairwise Balanced Designs.

Keywords: A Williamson type matrices, Pairwise Balanced Designs, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ -resolvable PBD, Amicable designs, seed vectors, difference vectors.

Introduction

We begin with the following definitions:

Circulant Matrices: A Circulant matrix A is a square matrix whose first row, say $(a_1, a_2, ..., a_n)$ is arbitrary, each of the remaining rows begins with the last entry of the previous row and other entries are obtained by shifting those of previous row one place to the right. It is written as A = Circ $(a_1, a_2, ..., a_n)$

Williamson type Matrices[1-4]: Four square matrices A, B, C, D of order n are called Williamson-type matrices if

$$AA^T + BB^T + CC^T + DD^T = 4nI_n.$$

A,B,C,D are pairwise amicable i.e. XY^T is symmetric, for all $X, Y \in \{A, B, C, D\}$

Pairwise Balanced Design (PBD)[5-8]: Let v, $k_1, k_2, k_3, k_4, \lambda$ be positive integers. Then a (v, $k_1, k_2, k_3, k_4, \lambda$) - PBD is an ordered pair (V, B) consisting of a v-set V and a family B of subsets (called blocks) of V whose cardinalities $\in \{k_1, k_2, k_3, k_4\}$. k_1, k_2, k_3, k_4 are called block sizes and λ is called index of the PBD.

(a_1, a_2, a_3, a_4) **-resolvable PBD**[9,10]: Consider a PBD (V, B) of index λ. Then an a-resolvable class is a subset of B which together contains every element of V exactly a time. A PBD (V, B) is (a_1, a_2, a_3, a_4) **-**resolvable if B can be partitioned into subfamilies B_1, B_2, B_3, B_4 such that B_i is a α_i, – resolution class, i = 1, 2, 3, 4.

The following definition has been motivated by amicable matrices[11]

Amicable designs: Let D_1 and D_2 be two block designs defined on the same ground set X = {1, 2,……, v} and having b blocks. Let N_1 and N_2 be v x b incidence matrices corresponding to same ordering of the set X = {1, 2,……, v} and blocks of D_1 and D_2 such that appropriate ordering of blocks of D_1 and D_2 such that

$$N_1 N_2^T = N_2 N_1^T$$

Then D_1 and D_2 will be called amicable designs.

Size vector of a PBD: If A = $\{a_{11}, a_{12}, \dots a_{1k1}\}$ (mod n), B = $\{b_{11}, b_{12}, \dots b_{1k2}\}$ (mod n), C = $\{c_{11}, c_{12}, \dots c_{1k3}\}$ (mod n) and D = $\{d_{11}, d_{12}, \dots d_{1k4}\}$ (mod n), are amicably resolvable PBD then A = $\{k_1, k_2, k_3, k_4\}$ will be called size vector of the PBD.

Difference Frequency vector (DF vector) of a set or block of a design: Let n be odd and $m = \frac{n-1}{2}$. Let B=$\{a_1, a_2, ..., a_n\}$ (mod n) be a set of integers mod n. We form the multiset of all positive differences less than or equal to mod n of the set B. Let the difference i occur f_i times (i = 1,2,….,m) in the multiset. Then $(f_1, f_2, \dots, f_m)$ will be called DF vector of the set or block of a design.

Method for generating certain Williamson type matrices through suitable Pairwise Balanced Design (PBD): Since there is no listing of PBD suitable for our construction, for the search of suitable PBD, we proceed as follows:

Steps: Let n (odd) be the order of the Williamson type matrix and $m = \frac{(n-1)}{2}$

Step 1: Express 4n as the sum of four odd squares i.e. $4n = n_1^2 + n_2^2 + n_3^2 + n_4^2$

Let $k_i = \frac{(n - n_i)}{2}$, i = 1, 2, 3, 4. Clearly, $k_i \le m$

Let (k_1, k_2, k_3, k_4) be the size vector of required amicably resolvable PBD.

Step 2: Generation of feasible initial blocks $S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4}$ of resolution classes of PBD.

If k_1 is even, generate all $\frac{k_1}{2}$ – subsets of {1, 2, ………., m}

Adjoin the numbers $n - a_i (i = 1, 2, ... \frac{k_1}{2})$ to each subset $(a_1, a_2, ..., a_{\frac{k_1}{2}})$ to form a k_1 - subsets S_{k_1} of {1, 2, ………, n-1}.

Lable the k_1 - subsets thus formed as $A_1, A_2, ..., A_{\sigma_1}$ where $\sigma_1 = \binom{m}{\frac{k_1}{2}}$ or $\binom{m}{\frac{k_1 - 1}{2}}$

If k_1 is odd, do the same to $\frac{k_1 - 1}{2}$ – subsets of {1, 2, ……..,m}and adjoin 0 to each subset.

Similarly, form the k_2-subsets and k_3-subsets of {1, 2, .., n - 1} and lable them by $B_1, B_2, .., B_{\sigma_2}$ and $C_1, C_2, .., C_{\sigma_3}$ where

$$\sigma_2 = \binom{m}{\frac{k_2}{2}} \text{ or } \binom{m}{\frac{k_2 - 1}{2}} \quad \sigma_3 = \binom{m}{\frac{k_3}{2}} \text{ or } \binom{m}{\frac{k_3 - 1}{2}}$$

Finally, form all k_4 - subsets of {1,2,……..,n-1} and lable them by $D_1, D_2, ..., D_{\sigma_4}$ where $\sigma_4 = \binom{m}{\frac{k_4}{2}}$ or $\binom{m}{\frac{k_4 - 1}{2}}$

Step 3: Form the DF vectors of all subsets formed in step 2. If A_i, B_j, C_k, D_l are the labels of $S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4}$, label their DF-vectors by a_i, b_j, c_k, d_l.

Step 4: Generation of Quadruple of feasible DF vectors for S_{k_i} (i = 1,2, 3, 4) by a 4 x m array.

Generate a 4 x m array with entries 0, 1, 2, …………, (m-2) with the following properties:

i. The column sum of the array is $k_1 + k_2 + k_3 + k_4 - n$

ii. The sum of i^{th} row is $\binom{k_i}{2}$, i = 1, 2, 3, 4

Step 5: To find a solution array: A 4 x m array formed in step 4 will give 4 Williamson type matrices if its r^{th} row (r = 1, 2, 3, 4) coincides with DF vectors of step 3, having lable a_i, b_j, c_k, d_l respectively for some I, j, k, l. Such an array will be column solution array.

Step 6: To find Williamson type matrices from solution array: Find the lables of rows of solution array through step 5. Let a_i, b_j, c_k, d_l be the lables. Identify the subsets S_{k1}, S_{k2}, S_{k3}, S_{k4}, with lables A_i, B_j, C_k, D_l assigned in step 3.

Let $v_i = \{^{-1 \text{ if } i \in S_K}_{+1 \text{ otherwise}}$

Let $v^{(k)} = (v_1, v_2, .., v_n)$ where $k \in \{k_1, k_2, k_3, k_4\}$

Let A, B, C be the circulant with first row $v^{(k_1)}, v^{(k_2)}$ and $v^{(k_3)}$ respectively and D be the back circulant with first row $v^{(k_4)}$. Then A, B, C, D are required Williamson type matrices of order n.

Remarks: i By exhaustive search based on the above method, we can obtain all Williamson matrices together with all symmetric Williamson type matrices A, B, C and D where A, B, C are circulant and D is back circulant.

Remarks: ii Generation of Quadruple of feasible DF-vectors needs a suitable algorithm. We illustrate the above method by the following example:

Example 15: Take n = 9 (odd) then m = 4.

Step 1: Let $4n = 36 = 1^2 + 1^2 + 3^2 + 5^2$

$\Rightarrow n_1 = 1, n_2 = 1, n_3 = 3, n_4 = 5$

and $k_1 = \frac{n - n_1}{2} = 4, k_2 = \frac{n - n_2}{2} = 4, k_3 = \frac{n - n_3}{2} = 3, k_4 = \frac{n - n_4}{2} = 2$

$\therefore (k_1, k_2, k_3, k_4) = (4, 4, 3, 2)$

is the size vector of suitable PBD.

Step 2 : $\because k_1 = 4$, is even. Generate all $\frac{k_1}{2}$ – subsets of {1, 2, 3, 4}.

First we find 2 - subsets of {1, 2, 3, 4} i.e. (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)

Adjoin the numbers $n - a_i$ $(i = 1, 2, ..., \frac{k_1}{2})$ to each subset $(a_1, a_2, ..., a_s)$ to form a k_1 - subsets of S_{K_1} of {1, 2, ..., 8}.

2-subsets	4-subsets	Lables
(1, 2)	(1, 2, 7, 8)	A_1
(1, 3)	(1, 3, 6, 8)	A_2
(1, 4)	(1, 4, 5, 8)	A_3
(2, 3)	(2, 3, 6, 7)	A_4
(2, 4)	(2, 4, 5, 7)	A_5
(3, 4)	(3, 4, 5, 6)	A_6

Similarly, we generate all $\frac{k_2}{2}-$ subsets and $\frac{k_4}{2}-$ subsets (since they are even)

$\frac{k_2}{2}-$ subsets

2-subsets	4-subsets	Lables
(1, 2)	(1, 2, 7, 8)	B_1
(1, 3)	(1, 3, 6, 8)	B_2
(1, 4)	(1, 4, 5, 8)	B_3
(2, 3)	(2, 3, 6, 7)	B_4
(2, 4)	(2, 4, 5, 7)	B_5
(3, 4)	(3, 4, 5, 6)	B_6

$\frac{k_4}{2}-$ subsets

1-subsets	2-subsets	Lables
(1)	(1, 8)	D_1
(2)	(2, 7)	D_2
(3)	(3, 6)	D_3
(4)	(4, 5)	D_4

And $\because k_3 = 3$, is odd. Generate all $\frac{k_3-1}{2}$ $-$ subsets of $\{1, 2, 3, 4\}$ and adjoin 0 to each subset

2-subsets	3-subsets	Lables
(0, 1)	(0, 1, 8)	C_1
(0, 2)	(0, 2, 7)	C_2
(0, 3)	(0, 3, 6)	C_3
(0, 4)	(0, 4, 5)	C_4

Step 3: We form the DF vectors of all subsets formed in step 2.

Lables	DF vectors	New lables
A_1	$(2,1,2,1)$	a_1
A_2	$(0,3,1,2)$	a_2
A_3	$(1,1,2,2)$	a_3
A_4	$(2,0,1,3)$	a_4
A_5	$(1,2,2,1)$	a_5
A_6	$(3,2,1,0)$	a_6
B_1	$(2,1,2,1)$	b_1
B_2	$(0,3,1,2)$	b_2
B_3	$(1,1,2,2)$	b_3
B_4	$(2,0,1,3)$	b_4
B_5	$(1,2,2,1)$	b_5
B_6	$(3,2,1,0)$	b_6
C_1	$(2,1,0,0)$	c_1
C_2	$(0,2,0,1)$	c_2
C_3	$(0,0,3,0)$	c_3
C_4	$(1,0,0,2)$	c_4
D_1	$(0,1,0,0)$	d_1
D_2	$(0,0,0,1)$	d_2
D_3	$(0,0,1,0)$	d_3
D_4	$(1,0,0,0)$	d_4

Step 4: Then we generate quadruple of feasible DF vectors for S_{k_i} $(i = 1,2,3,4)$ by a $4\times m$ array. If we take A_1, B_1, C_2 and D_2 then we get 4×4 array as

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ i.e. } \begin{bmatrix} a_1 \\ b_1 \\ c_2 \\ d_2 \end{bmatrix} \text{where column sum is } 4 = k_1 + k_2 + k_3 + k_4 - n.$$

and sum of i^{th} row is $\binom{k_i}{2}$, $i = 1, 2, 3, 4$.

Step 5 : Now we find a solution array. Clearly, (a_1, b_1, c_2, d_2) is the required solution array.

Step 6 : Finally we find Williamson type matrices from the above solution array.

$\because (a_1, b_1, c_2, d_2)$ is the required solution array

$\therefore$ we can select $S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4}$ with lables A_1, B_1, C_2, D_2.

Let $v_i = \{ {}^{-1 \text{ if } i \in S_K}_{+1 \text{ otherwise}}$

Let $v^{(k)} = (v_1, v_2, \ldots\ldots\ldots \ldots\ldots, v_n)$ where $k \in \{k_1, k_2, k_3, k_4\}$

$v^{(k_1)} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9) =$
$(-1, -1, +1, +1, +1, +1, -1, -1, +1)$

Similarly, $v^{(k_2)} = (-1, -1, +1, +1, +1, +1, -1, -1, +1)$

$v^{(k_3)} = (+1, -1, +1, +1, +1, +1, -1, +1, -1)$

$v^{(k_4)} = (+1, -1, +1, +1, +1, +1, -1, +1, +1)$

Then we can get the required Williamson type matrices A, B, C, D where A, B, C be the circulant matrices with first row $v^{(k_1)}, v^{(k_2)}, v^{(k_3)}$ respectively and D be the back circulant with first row $v^{(k_4)}$.

Thus, from the (suitable) PBD (1, 2, 7, 8) (1, 2, 7, 8) (0, 2, 7) (2, 7) (mod n), we get the Williamson type matrices

A = circ (-1, -1, +1, +1, +1, +1, -1, -1, +1)

B = circ (-1, -1, +1, +1, +1, +1, -1, -1, +1)

C = circ (+1, -1, +1, +1, +1, +1, -1, +1, −1)

D = backcirc (+1, -1, +1, +1, +1, +1, -1, +1, +1).

Methodology

We have established the results with the help of amicable designs and techniques of Linear Algebra.

Results and Discussion

Williamson type matrices which helps in the construction of Hadamard matrices also helps in the construction of (α_1, α_2,…., α_n)- amicably resolvable Pairwise balanced designs. In this paper we have shown that the converse of the above result is also hold. The family of (α_1, α_2,…., α_n)- amicably resolvable Pairwise balanced designs is a new concept in the theory of block designs.

Conclusion

A new type of non-circulant symmetric Williamson matrices are constructed through a new family of Pairwise balanced designs.

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