Research Journal of Mathematical and Statistical Sciences _____________________________ E-ISSN 2320–6047
Vol. **4(2),** 6-12, March **(2016)** *Res. J. Mathematical and Statistical Sci.*

Critical Evaluation of Four Differencing Schemes for A Steady Convection-Diffusion Problem

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Available online at: www.isca.in, www.isca.me
Received 6th February 2016, revised 28th February 2016, accepted 10th March 2016

Abstract

A steady convection diffusion problem is taken to compare the behavior and accuracy of four discretization schemes namely, Central Differencing Scheme, Upwind Differencing scheme, Hybrid Differencing Scheme and QUICK scheme. This well-known problem is solved numerically and solutions are discussed graphically. It is known that false diffusion arises in multidimensional flow problems only. Hence discretization errors may be investigated only in one-dimensional problems. Thus, the model taken to compare four schemes is a one dimensional flow model. It is validated that central differencing scheme give fairly good results for small Peclet number only, whereas upwind differencing scheme may be used for both large and small Peclet number. Hybrid scheme gives better results than QUICK scheme which works very well with large Peclet number but not for small Peclet number.

Keywords: Convection –diffusion Problem, Central Differencing Scheme, Upwind Differencing scheme, Hybrid Differencing Scheme, QUICK scheme, Peclet Number.

Introduction

In many applications of applied sciences and engineering we frequently arrive at Convection-diffusion problems. The presence of the convection term makes the discretization of the equation difficult which results in inconvenience to work out the accurate numerical solution of the convection diffusion problem. For convection-dominated problems we need to apply special techniques so as to get stable and bounded solution. Better schemes to are still required to approximate the convection term and substantial research has been directed towards finding ideal discretisation schemes.

There are many schemes to solve convection-diffusion equation. Many of the schemes are given in the classic text by Patankar[1] Ferziger and Peric give a contemporary discussion of finite volume methods for convection diffusion problem[2]. Versteeg and Malalasekera provide a thorough discussion of many discretization schemes, but they do not discuss the case of non-uniform meshes[3]. Majumdar provide a basic analysis of convection modeling of the one and two dimensional convection-diffusion equation[4]. Wesseling discusses different approximations to the convective terms and also strongly investigates finite volume method[5]. Compararitive and critical studies of various methods are done by Patel et al., Lazarov et al. Shukla et al., Stynes etc[6-11].

In this paper we discuss and compare the Central Difference Scheme, Upwind Differencing Scheme, Hybrid Differencing Scheme and QUICK Scheme for a steady one dimensional convection-diffusion problem. The one-dimensional convection-diffusion equation is a compact model of transport of heat, mass and other passive scalars. By applying the finite volume method to this equation we are open to use different schemes for approximating the convection term. We are considering only four schemes and the analysis only involves one model equation.

Discretization Algorithm

In the absence of sources, steady convection and diffusion of a property ϕ in a one dimensional domain is sketched in figure-1. The governing equation is

$$\frac{d}{dx}(\rho\, u\, \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) \tag{1}$$

which on integration over the control volume of fig (1) gives

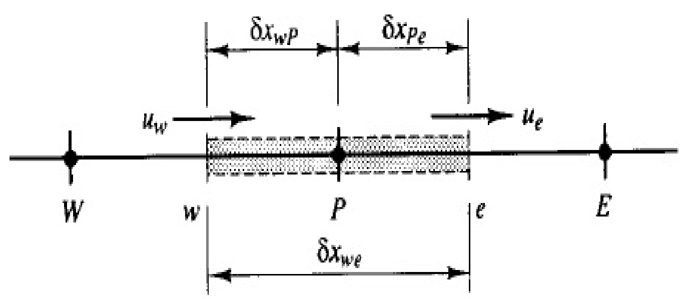

Figure-1
Control volume for one-dimensional problem

$$(\rho uA\phi)_e - (\rho uA\phi)_w = \left(\Gamma A \frac{d\phi}{dx}\right)_e - \left(\Gamma A \frac{d\phi}{dx}\right)_w \quad (2)$$

The flow satisfy the continuity. Therefore

$$\frac{d(\rho u)}{dx} = 0 \quad (3)$$

Integration of continuity equation gives

$$(\rho uA)_e - (\rho uA)_w = 0 \quad (4)$$

For convenience, we take

$$F = \rho u \text{ and } D = \frac{\Gamma}{\delta x}$$

And define cell Peclet number as

$$P_e = \frac{F}{D} \quad (5)$$

which is a dimensionless number which is significant in the study of transport phenomena. Now the values of the variables *F* and *D* at both the cell faces can be written as

$$F_e = (\rho u)_e \ , \ F_w = (\rho u)_w \text{ and}$$

$$D_e = \left(\frac{\Gamma_e}{\delta x_{PE}}\right) \ , D_w = \left(\frac{\Gamma_w}{\delta x_{WP}}\right)$$

To write down discretised equations, we need to approximate the terms in Equation-2
Assuming

$$A_w = A_e = A$$

The convection diffusion Equation-2 can be written as

$$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) \quad (6)$$

and the integrated Equation of continuity (4) as

$$F_e - F_w = 0$$

Central Differencing Scheme (CDS): The cell face value of property ϕ for a uniform grid, can be written as

$$\phi_e = (\phi_P + \phi_E)/2$$

$$\phi_w = (\phi_W + \phi_P)/2$$

Substitution of the above expressions into the convection term of Equation-6 yields

$$\frac{F_e}{2}(\phi_P + \phi_E) - \frac{F_w}{2}(\phi_W + \phi_P) = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

This can be rearranged to give the standard form as

$$a_P\phi_P = a_W\phi_W + a_E\phi_E \quad (7)$$

where:

a_W	a_E	a_P
$D_w + \frac{F_w}{2}$	$D_e - \frac{F_e}{2}$	$a_W + a_E + (F_e - F_w)$

Upwind Differencing Scheme (UDS): The upwind differencing scheme considers the direction of the flow, which lacks in the central differencing scheme. When the flow is in positive direction $(F_w > 0, F_e > 0)$ the cell face values are calculated using nodal values shown in figure-(2)

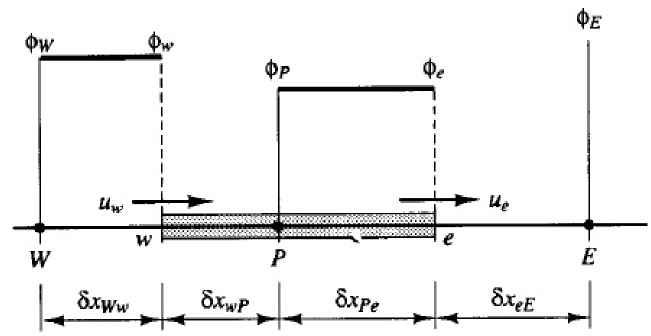

Figure-2
Upwind scheme for positive flow direction

The upwind scheme sets

$$\phi_w = \phi_W \ , \ \phi_e = \phi_P$$

And the discretised convection diffusion equation given by Eqution-6 after rearrangement gives

$$\left[(D_w + F_w) + D_e + (F_e - F_w)\right]\phi_P = (D_w + F_w)\phi_W + D_e\phi_E \quad (8)$$

Equation-8 can be written in usual general form as

$$a_P\phi_P = a_W\phi_W + a_E\phi_E \quad (9)$$

where:

	a_W	a_E	a_P
$F_e > 0$, $F_w > 0$	$D_w + F_w$	D_e	$a_W + a_E + (F_e - F_w)$
$F_e < 0$, $F_w < 0$	D_w	$D_e - F_e$	$a_W + a_E + (F_e - F_w)$

Hybrid Differencing Scheme (HDS): The basis of Hybrid Differencing scheme is the combination of Central differencing and Upwind Differencing Schemes. It employs Central Differencing Scheme for small Peclet number ($P_e \le 2$) and Upwind scheme for large Peclet number ($P_e \ge 2$). The general form of discretised equation in Hybrid differencing scheme is

$$a_P\phi_P = a_W\phi_W + a_E\phi_E \quad (10)$$

where

a_W	a_E	a_P
$\max\left[F_w,\left(D_w+\frac{F_w}{2}\right),0\right]$	$\max\left[-F_e,\left(D_e-\frac{F_e}{2}\right),0\right]$	$a_W+a_E+(F_e-F_w)$

Quadratic Upwind Differencing Scheme (QUICK): For cell face values, this scheme uses a three point upstream weighted quadratic interpolation. The general form of discretised equation in QUICK scheme is

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_{WW}\phi_{WW} + a_{EE}\phi_{EE} \qquad (11)$$

Where

$$a_P = a_W + a_E + a_{EE} + a_{WW} + (F_e - F_w)$$

a_W	a_{WW}	a_E	a_{EE}
$D_w+\frac{6}{8}\alpha_w F_w+\frac{1}{8}\alpha_e F_e + \frac{3}{8}(1-\alpha_w)F_w$	$-\frac{1}{8}\alpha_w F_w$	$D_e-\frac{3}{8}\alpha_e F_e-\frac{6}{8}(1-\alpha_e)F_e + \frac{1}{8}(1-\alpha_w)F_w$	$\frac{1}{8}(1-\alpha_e)F_e$

and

$$\alpha_w = 1 \text{ for } F_w > 0 \text{ and } \alpha_e = 1 \text{ for } F_e > 0$$
$$\alpha_w = 0 \text{ for } F_w < 0 \text{ and } \alpha_e = 0 \text{ for } F_e < 0$$

Convection and Diffusion Equation: Numerical Results

Let us assume the convection and diffusion of the property ϕ through a one dimensional domain shown by figure-3.

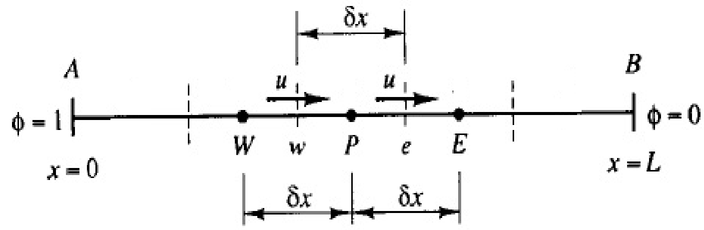

Figure-3
The discretization domain

The governing equation is given by Equation. (1) and the boundary conditions are ϕ_A=1 and ϕ_B=0. Other data to be used are length L= 1.0 m, ρ =1.0 Kg/m^3, Γ = 0.1 Kg/m.s. The domain is divided into ten equal control volumes, so we have δx=0.1 m. The analytical solution to Eqn.(1) is

$$\frac{\phi-\phi_0}{\phi_L-\phi_0} = \frac{\exp(\rho u x/\Gamma)-1}{\exp(\rho u L/\Gamma)-1} \qquad (12)$$

We will discuss and compare the schemes with the analytical result for different values of cell Peclet number by taking different values of flow velocity *u*.

The comparison of the analytical solution (AS) with the solution;ns obtained using Central Differencing Scheme, Upwind Differencing Schemes, Hybrid Differencing Scheme and QUICK scheme is given in Table 1,2 and 3 for different values of Peclet number.

Table-1
Comparison of results obtained using different schemes with the analytical solution for u=0.1 m/s and P_e = 0.1

Node	Central Differencing Scheme	Upwind Differencing Scheme	Hybrid Differencing Scheme	QUICK Scheme	Analytical solution (AS)	Error in CDS	Error in UDS	Error in HDS	Error in QUICK
0.05	0.9709	0.9687	1.00	0.9389	0.9702	- 0.0007	0.0015	- 0.0298	0.0313
0.15	0.9067	0.9031	1.00	0.8072	0.9058	0.0027	0.0027	- 0.0942	0.0986
0.25	0.8356	0.8309	1.00	0.6616	0.8347	0.0038	0.0038	- 0.1653	0.1731
0.35	0.7572	0.7515	1.00	0.5268	0.7561	0.0046	0.0046	- 0.2439	0.2293
0.45	0.6704	0.6641	1.00	0.4041	0.6693	0.0052	0.0052	- 0.3307	0.2652
0.55	0.5746	0.5680	1.00	0.2947	0.5733	0.0053	0.0053	- 0.4267	0.2786
0.65	0.4686	0.4623	1.00	0.2002	0.4672	0.0049	0.0049	- 0.5328	0.2670
0.75	0.3515	0.3460	1.00	0.1219	0.3499	0.0039	0.0039	- 0.6501	0.2280
0.85	0.2221	0.2181	1.00	0.0618	0.2204	0.0023	0.0023	- 0.7796	0.1586
0.95	0.0790	0.0774	0.0476	0.0216	0.0772	- 0.0002	- 0.0002	0.0476	0.0556
Abs. error						0.0336	0.0344	3.2827	1.7853

Table-2
Comparison of results obtained using different schemes with the analytical solution for *u*= 3 *m/s* and P_e = 3

Node	Central Differencing Scheme	Upwind Differencing Scheme	Hybrid Differencing Scheme	QUICK Scheme	Analytical solution (AS)	Error in CDS	Error in UDS	Error in HDS	Error in QUICK
0.05	1	1	1	1	1	0	0	0	0
0.15	1	1	1	1	1	0	0	0	0
0.25	1	1	1	1	1	0	0	0	0
0.35	1	0.9999	1	1	1	0	0.0001	0	0
0.45	0.9998	0.9996	1	1	1	0.0002	0.0004	0	0
0.55	1.0008	0.9984	1	1	1	- 0.0008	0.0016	0	0
0.65	0.9960	0.9938	1	1	1	0.0040	0.0062	0	0
0.75	1.0200	0.9750	1	1	0.9994	-0.0206	0.0244	-0.0006	-0.0006
0.85	0.9000	0.9000	1	1	0.9889	0.0889	0.0889	-0.0111	-0.0111
0.95	1.5000	0.6000	0.6000	1	0.7769	-0.7231	0.1769	0.1769	-0.2231
Abs error						0.8376	0.2923	0.1886	0.2348

Table-3
Comparison of results obtained using different schemes with the analytical solution for *u*=10 *m/s* and P_e = 10

Node	Central Differencing Scheme	Upwind Differencing Scheme	Hybrid Differencing Scheme	QUICK Scheme	Analytical solution (AS)	Error in CDS	Error in UDS	Error in HDS	Error in QUICK
0.05	0.9117	1	1	0.9698	1.00	0.0883	0	0	0.0302
0.15	1.1760	1	1	0.9615	1.00	-0.1760	0	0	0.0385
0.25	0.7794	1	1	0.9599	1.00	0.2206	0	0	0.0401
0.35	1.3750	0.9999	1	0.9609	1.00	-0.3750	0.0001	0	0.0391
0.45	0.4816	0.9999	1	0.9573	1.00	0.5184	0.0001	0	0.0427
0.55	1.8217	0.9999	1	0.9686	1.00	-0.8217	0.0001	0	0.0314
0.65	-0.1884	0.9998	1	0.9322	1.00	1.1884	0.0002	0	0.0678
0.75	2.8267	0.9986	1	1.0498	1.00	-1.8267	0.0014	0	-0.0498
0.85	-1.6960	0.9848	1	0.6700	1.00	2.6960	0.0152	0	0.33
0.95	5.0880	0.8333	0.8333	1.8973	0.9933	-4.0880	0.16	0.16	-0.904
Abs. error						11.9991	0.1771	0.16	1.9255

Graphical Representation of Convergence: The graphical representation of solutions obtained in Table-1,2 and 3 are shown in figure-4,5 and 6 respectively for different values of Peclet number.

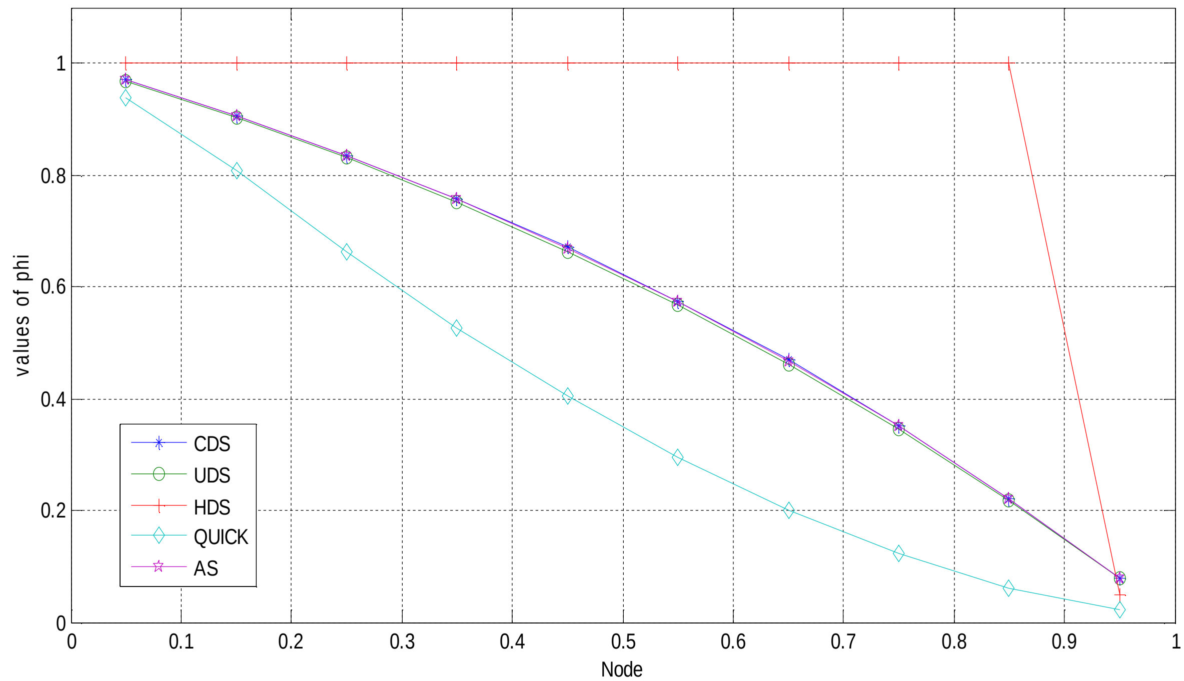

Figure-4
Comparison of results obtained using different schemes with the analytical solution for *u*=0.1 *m/s* and P_e = 0.1

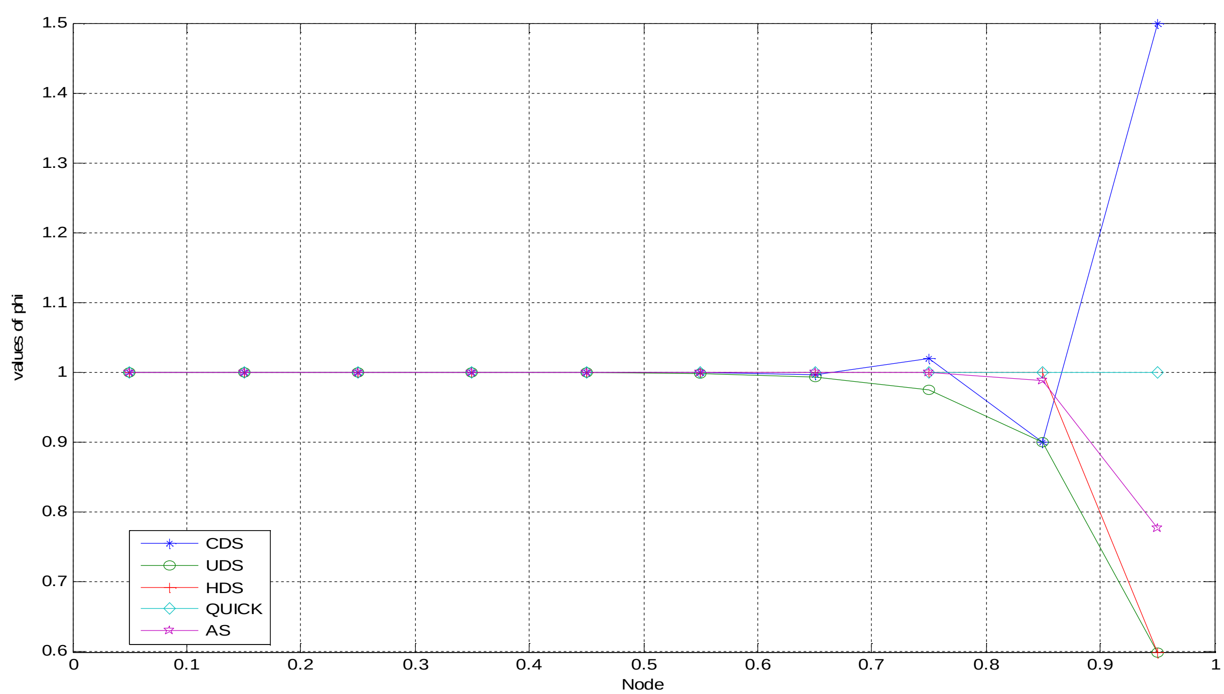

Figure-5
Comparison of results obtained using different schemes with the analytical solution for *u*= 3 *m/s* and P_e = 3

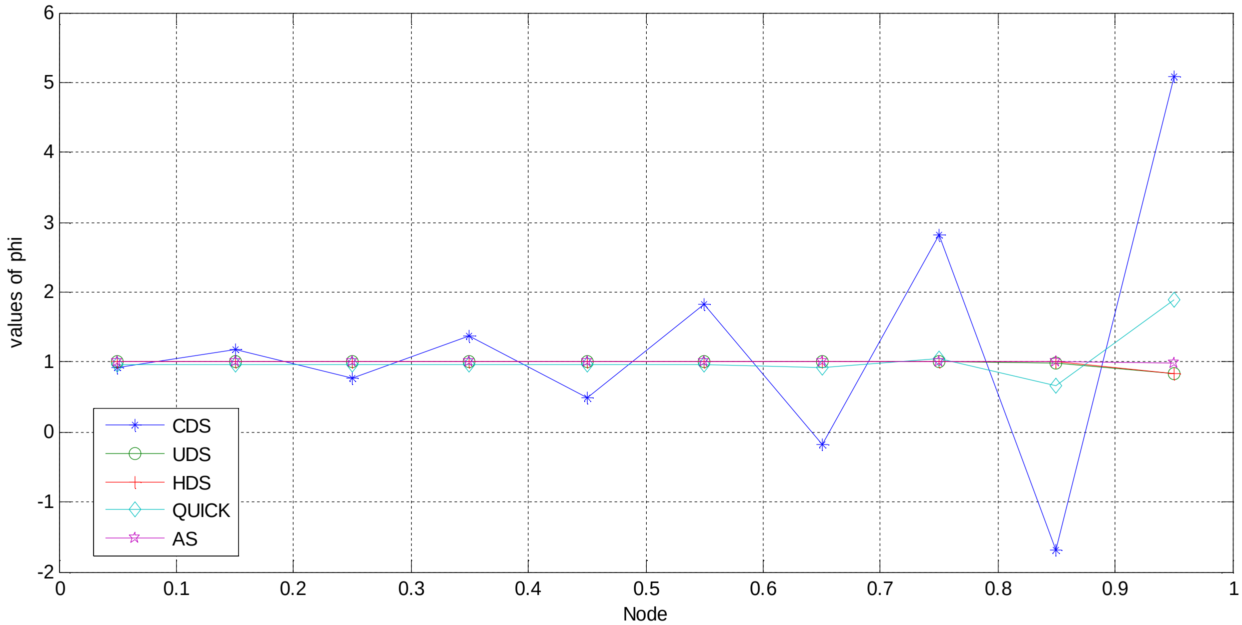

Figure-6
Comparison of results obtained using different schemes with the analytical solution for *u*=10 *m/s* and P_e = 10 .The oscillatory numerical solution of Central Difference Scheme shown are due to the existence of negative a_E or a_W

Discussion

In present study, four discretization schemes, namely Central Differencing Scheme, Upwind differencing Scheme, Hybrid Differencing Scheme and QUICK Schemes are compared for one dimensional steady state convection diffusion problem. The results of computations presented in this paper corroborate the following conclusions.

For small values of P_e both Central Differencing Scheme and Upwind Differencing Schemes give results that are close to the analytical solutions.

The appearance of 'wiggles' begins at P_e =3, in Central Differencing solutions. At this value of cell Peclet number, the east coefficient in the scheme becomes negative which violates the requirement of boundedness and lead to unrealistic solutions as shown in Table-1 and 3 where the solutions even lie outside the range established by the boundary values. All other Schemes gives very accurate and realistic results for P_e =3.

For P_e =10, Hybrid differencing scheme gives same results as exact solution except near boundary B.

Numerical solutions to Equation-1 attained with the upwind difference scheme and hybrid difference scheme never oscillate for any value of P_e whereas solutions obtained with the central difference scheme on a uniform mesh will oscillate if $P_e > 2$.

Conclusion

There are a lot of schemes to compute the numerical solution of the convective-diffusion problems and similar problems. In their paper, Wang and Hutter compare at least twelve methods for the discretization of convection diffusion problem[12]. The central difference scheme seems to yield accurate results for low cell Peclet number, but for $P_e > 2$, the scheme produces a solution that seems to oscillate about the exact solution. By refining the meshes, one can reduce the cell Peclet number and overcome the problem of oscillations. As one of the aim of numerical modeling is to obtain solutions which are mesh-independent, one can opt for a simple modification of Central Differencing Scheme to Upwind Differencing scheme. The Upwind scheme considers the direction of the flow but its accuracy is only first order on the basis of Taylor series truncation error. Due to this, diffusion errors arise in the solution of multi dimensional problems, when the flow is not aligned with the grid lines. So, this modification is not entirely suitable for accurate flow calculations. The hybrid difference scheme of Spalding (1972) exploits the favorable properties of upwind and central difference schemes and gives physically realistic solutions. It is highly stable, but it also has only first order accuracy. To retain the accuracy, higher order discretisation schemes may be employed. The QUICK scheme of Leonard (1979) has third order accuracy in terms of Taylor's series truncation error. It is conservative also, but in some cases it may give unstable and unbounded solutions. The QUICK scheme is therefore conditionally stable. However, if it is used carefully, QUICK scheme can give very accurate results.

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