



Some Identities Involving Generalized Left (θ, ϕ) -Derivations in Semiprime Rings

Mondal Sachhidananda

Shyamsundarpur High School, P. O.- Hanschara, Dist: Purba Medinipur, 721625, INDIA

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Abstract

Let us suppose that R be a semiprime ring with two epimorphisms θ, ϕ and I a nonzero ideal of R . By generalized left (θ, ϕ) -derivation of R we mean an additive mapping $F : R \rightarrow R$ such that $F(ab) = \theta(a)F(b) + \phi(b)d(a)$ holds for all $a, b \in R$, where $d : R \rightarrow R$ is a left (θ, ϕ) -derivation of R . The goal of the present paper is to study the following identities: (i) $F([a, b]) = \theta([a, b])$, (ii) $F(aob) = \theta(aob)$, (iii) $F([a, b]) = \theta(aob)$, (iv) $F(aob) = \theta([a, b])$, (v) $F(ab) \pm \theta(ab) \in Z(R)$, (vi) $F(ab) \pm \theta(ba) \in Z(R)$, for all $a, b \in I$. Mathematics Subject Classification 2010. 16W25, 16W80, 16N60.

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Introduction

In the present article, throughout, by R we mean an associative ring having $Z(R)$ as center of R . For any $a, b \in R$ $[a, b]$ represents the commutator $ab-ba$ and the anti-commutator $ab+ba$ is represented by the symbol (aob) . If for any $x, y \in R$, $xRy=0$ implies either $x=0$ or $y=0$ in R then we say the ring R is prime and if for any $x \in R$, $xRx=0$ implies $x=0$ then the ring is said to be semiprime. By derivation we mean an additive mapping $d : R \rightarrow R$ such that $d(ab)=d(a)b + ad(b)$ holds for all $a, b \in R$. An additive mapping $d : R \rightarrow R$ is said to be a left derivation if $d(ab)= ad(b) + bd(a)$ holds for all $a, b \in R$. For any two endomorphisms θ, ϕ an additive mapping $d : R \rightarrow R$ is called a left (θ, ϕ) -derivation, of R , if $d(ab)= \theta(a)d(b)+ \phi(b)d(a)$ holds for all $a, b \in R$. The mapping is called a left (θ, θ) -derivation if we take $\phi=\theta$. By generalized left (θ, ϕ) -derivation of R we mean an additive mapping $F : R \rightarrow R$ such that $F(ab) = \theta(a)F(b) + \phi(b)d(a)$ holds for all $a, b \in R$, where $d : R \rightarrow R$ is a left (θ, ϕ) -derivation of R . When we take $\phi = \theta$, then we say that F is generalized left (θ, θ) -derivation of R corresponding to a left (θ, θ) -derivation d of R . For a subset S of R , a mapping $f : R \rightarrow R$ is called commuting on S , if $[f(a), a]=0$ for all $a \in S$ and if $[f(a), a] \in Z(R)$ for all $a \in S$ then f is known as centralizing on S .

The relationship between some specific types of derivations and the commutativity of the ring R was initiated by Posner¹, who proved that if a prime ring R admits a nonzero centralizing derivation then it must be commutative. After this result, several authors have investigated in this specific field of algebra and found many valuable results.

The concept of left derivation was initiated by Bresar and Vukman² and it was shown that if a prime ring R of characteristic different from 2 and 3 must be commutative

when it admits a nonzero Jordan left derivation. Ashraf and Ali³ introduced the concepts of generalized left derivation and generalized Jordan left derivation. Xu and Zhang⁴ describe generalized left (θ, ϕ) -derivations in prime rings, and proved that an additive mapping in a ring R acting as a homomorphism or antihomomorphism on an additive subgroup S of R must be either a mapping acting as a homomorphism on S or a mapping acting as an antihomomorphism on S . Ashraf et. al⁵. proved for prime ring R if $G : R \rightarrow R$ is a generalized left (α, β) -derivation associated with a left (α, β) -derivation δ of R such that G acts as homomorphism or antihomomorphism on a nonzero ideal I of R , then either R is commutative or $\delta = 0$ on R . in this paper, authors considered α and β as automorphisms of R .

Motivated by these above results, we study the following identities in semiprime ring R :

(i) $F([a, b]) = \theta([a, b])$, (ii) $F(aob) = \theta(aob)$, (iii) $F([a, b]) = \theta(aob)$, (iv) $F(aob) = \theta([a, b])$, (v) $F(ab) \pm \theta(ab) \in Z(R)$, (vi) $F(ab) \pm \theta(ba) \in Z(R)$, for all $a, b \in I$. where I is an ideal of R and θ, ϕ be two epimorphisms of R and F is a generalized left (θ, ϕ) -derivation of R associated with a left (θ, ϕ) -derivation d of R .

We need the following facts which will be used to prove our Theorems.

Fact-1. If $xIy = 0$, for a nonzero ideal I of a prime ring R and $x, y \in R$, then either $x=0$ or $y=0$.

Fact-2. (a) Center of a nonzero one-sided ideal of a semiprime ring R is contained in $Z(R)$. If I be a commutative one-sided ideal of the ring R then $I \subseteq Z(R)$ ⁶. ([Lemma 2])

(b) Let R a prime ring and I be a nonzero ideal of R , such that $I \subseteq Z(R)$ then R must be commutative.

Fact-3. If I be a nonzero ideal of any ring R and θ an epimorphism of R , then $\theta(I)$ is an ideal of R .

Fact-4. Let I be a nonzero ideal of a prime ring R and θ, φ two epimorphisms of R such that $\theta(I) \neq 0$ or $\varphi(I) \neq 0$. Let us consider a left (θ, φ) -derivation $\delta: R \rightarrow R$ of R . If $\delta(a)=0$ holds for all $a \in I$, then $\delta(R)=0$.

Proof. By our hypothesis, we have $0 = \delta(xr) = \theta(x)\delta(r) + \varphi(r)\delta(x) = \theta(x)\delta(r)$ for all $a \in I$ and $r \in R$, which gives $\theta(I)\delta(R)=0$. Since $\theta(I)$ is a nonzero ideal of R and R is prime, it follows that $\delta(R)=0$. On the other hand, $0 = \delta(rx) = \theta(r)\delta(x) + \varphi(x)\delta(r) = \varphi(x)\delta(r)$ for all $a \in I$ and $r \in R$, that is $\varphi(I)\delta(R)=0$. By the primeness of R and as $\varphi(I)$ is a nonzero ideal of R it follows that $\delta(R)=0$.

Fact-5. If J is an ideal of a semiprime ring R , then $J \cap \text{ann}_R(J) = 0$ (see [7, Corollary 2]).

Fact-6. For any prime ring R , $b \in R$ and $0 \neq x \in Z(R)$ and $bx \in Z(R)$ implies $b \in Z(R)$.

We present an example of generalized left (θ, φ) -derivation at the beginning.

Example 1.1. Let us suppose that $R = \left\{ \begin{pmatrix} m & n \\ 0 & m \end{pmatrix} \mid m, n \in \mathbb{Z} \right\}$ where \mathbb{Z} is the set of all integers. Obviously, R is not semiprime. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$. Let us define mappings $F, d, \theta, \varphi: R \rightarrow R$ by

$$F \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}, \quad d \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix}, \quad \theta \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}, \quad \text{and } \tau \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 2b \\ 0 & a \end{pmatrix}$$

for all $a, b \in \mathbb{Z}$.

It is easy to see that θ and φ are two epimorphisms of R and d be a left (θ, φ) -derivation of R . Now $F(r_1 r_2) = \theta(r_1)F(r_2) + \varphi(r_2)d(r_1)$ holds for all $r_1, r_2 \in R$, that is, F is a generalized left (θ, φ) -derivation of R corresponding to the left (θ, φ) -derivation d .

Main Results

Lemma 2.1. Let us consider a semiprime ring R , with two epimorphisms θ, φ and a nonzero ideal I of R . Let us consider a left (θ, φ) -derivation d of R such that $\varphi(I)d(I) \neq 0$. If $[R, \varphi(a)]\varphi(I)d(a) = 0$ for all $a \in I$, then a nonzero central ideal contained in R .

Proof: From our hypothesis

$$[R, \varphi(a)]\varphi(I)d(a) = 0 \tag{1}$$

for all $a \in I$. By using semiprimeness of R , R contain a family $P = \{P_i \mid i \in \mathbb{A}\}$ of prime ideals such that $\bigcap P_i = \{0\}$. For a typical member P_i of P and $a \in I$, it follows that

$$[R, \varphi(a)] \subseteq P_i \text{ or } \varphi(I)d(a) \subseteq P_i.$$

For fixed P_i , the set of elements $a \in I$ are subgroups of I , which are additive and whose union is I , for which these two conditions hold; therefore,

$$[R, \varphi(I)] \subseteq P_i \text{ or } \varphi(I)d(I) \subseteq P_i.$$

Both the cases together implies $[R, \varphi(I)]\varphi(I)d(I) \subseteq P_i$ for any $P_i \in P$. That is,

$$[R, \varphi(I)]\varphi(I)d(I) \subseteq \bigcap_{a \in \mathbb{A}} P_i = 0, \text{ thus } [R, \varphi(I)]\varphi(I)d(I) = 0.$$

Thus $0 = [R, \varphi(RIR)]\varphi(RI)d(I) = [R, R\varphi(I)R]R\varphi(I)d(I)$ and so $0 = [R, R\varphi(I)d(I)R]R\varphi(I)d(I)R$. This implies $0 = [R, K]RK$, where $K = R\varphi(I)d(I)R$ is a nonzero ideal of R , since $\varphi(I)d(I) \neq 0$. Then $0 = [R, K]R[R, K]$. Using semiprimeness of R , it is found that $0 = [R, K]$ that is $K \subseteq Z(R)$. Hence the proof is completed.

Lemma 2.2. Let R be a semiprime ring with a nonzero left (θ, θ) -derivation $d: R \rightarrow R$ of R , where θ be an epimorphisms of R such that $\theta(I)d(I) \neq 0$ for any nonzero ideal I of R . If for all $a \in I$, $[d(a), \theta(a)] = 0$ holds, then a nonzero central ideal contained in R . Also R is commutative when it is a prime ring.

Proof: By our assumption we have

$$[d(a), \theta(a)] = 0 \tag{2}$$

for all $a \in I$. Linearizing it we get

$$[d(a), \theta(b)] + [d(b), \theta(a)] = 0 \tag{3}$$

for all $a, b \in I$. Putting ab instead of b in (3) to get

$$\theta(a)[d(a), \theta(b)] + [d(a), \theta(a)]\theta(b) + \theta(a)[d(b), \theta(a)] + \theta(b)[d(a), \theta(a)] + [\theta(b), \theta(a)]d(a) = 0 \tag{4}$$

holds for all $a, b \in I$. Hence by using (2) and (3) the above relation reduces to $[\theta(b), \theta(a)]d(a) = 0$ for all $a, b \in I$. Putting tb for b , for $t \in R$, which gives $[\theta(t), \theta(a)]\theta(b)d(a) = 0$ for all $t \in R$ and $a, b \in I$. Since θ is an epimorphism we find that $[R, \theta(a)]\theta(I)d(a) = 0$ for all $a \in I$. By using Lemma 2.1 we find that a nonzero central ideal contained in R . By Fact-2(b), as a nonzero central ideal contained in R , we get R is commutative when R is prime ring.

Let us start our discussions with the final results.

Theorem 2.3. Suppose R is a semiprime ring, θ and φ are two epimorphisms of R and a nonzero ideal I of R . Consider a

generalized left (θ, φ) -derivation F of R associated with a left (θ, φ) -derivation d of R such that $\varphi(I)d(I) \neq 0$. If $F([a, b]) = \theta([a, b])$ holds for all $a, b \in I$, then a nonzero central ideal contained in R . Moreover, when R is prime then R is commutative.

Proof: Let us start with the case $F \neq 0$. Then by our hypothesis

$$F([a, b]) = \theta([a, b]) \quad (5)$$

for all $a, b \in I$. Replacing a with ab in (5), we get

$$F(b[a, b]) = \theta(b)F([a, b]) + \varphi([a, b])d(b) = \theta(b)\theta([a, b]), \quad (5)$$

which gives by (5) that

$$[\varphi(a), \varphi(b)]d(b) = 0 \quad (6)$$

for all $a, b \in I$. Again, substituting ta for a in (7), for $t \in R$, we obtained that

$$[\varphi(t), \varphi(b)]\varphi(a)d(b) = 0 \quad (8)$$

for all $a, b \in I$. As φ is an epimorphism, for all $b \in I$, this gives $[R, \varphi(b)]\varphi(I)d(b) = 0$. Hence using Lemma 2.1, we can find a nonzero central ideal in R .

Next we assume the case $F = 0$, in this case, $\theta([a, b]) = 0$ for all $a, b \in I$, that is $[\theta(I), \theta(I)] = 0$.

Thus $[\chi, \chi] = 0$, for a nonzero ideal $\theta(I) = \chi$ of R . Hence, $0 = [\chi, \chi] = [\chi R, \chi] = \chi[R, \chi] + [\chi, \chi]R = \chi[R, \chi] = \chi R[R, \chi]$. Then $[R, \chi]R[R, \chi] = 0$. By the semiprimeness of R , this implies $[R, \chi] = 0$. Again $[R, \chi] = 0$ implies χ is a central ideal of R which is also nonzero.

Now, when R is prime, then we can conclude that R is commutative, by Fact-2(b).

Theorem 2.4. Suppose R is a semiprime ring, θ and φ are two epimorphisms of R and a nonzero ideal I of R . Consider a generalized left (θ, φ) -derivation F of R associated with a left (θ, φ) -derivation d of R such that $\varphi(I)d(I) \neq 0$. If for all $a, b \in I$, $F(aob) = \theta(aob)$ holds, then a nonzero central ideal contained in R .

Moreover, when R is prime then R must be commutative (in this case $\text{char}(R) = 2$).

Proof: Let us start with the case $F \neq 0$. Then our hypothesis we gives for all $a, b \in I$ that,

$$F(aob) = \theta(aob) \quad (7)$$

Replacing ba in place of a in (9), we have for all $a, b \in I$,

$$F(b(aob)) = \theta(b)F(aob) + \varphi(aob)d(b) = \theta(b)\theta(aob) \quad (8)$$

This implies by (9), for all $a, b \in I$ that

$$\varphi(aob)d(b) = 0 \quad (9)$$

Again, changing a by ta in (11), for $t \in R$, we get $\varphi(t(aob)) - [t, b]a d(b) = 0$ for all $a, b \in I$, that is, $\varphi(t)\varphi(aob)d(b) - [\varphi(t), \varphi(b)]\varphi(a)d(b) = 0$ for all $a, b \in I$ and $t \in R$. In view of (11), above relation gives,

$$[\varphi(t), \varphi(b)]\varphi(a)d(b) = 0 \quad (10)$$

for all $a, b \in I$ and $t \in R$. This relation is identical with (8) in Theorem 2.3, and so by the same argument in Theorem 2.3, we have the required result.

Next we consider the case $F = 0$. Then we have for all $a, b \in I$, that $\theta(aob) = 0$ which implies $(\theta(I) \circ \theta(I)) = 0$. Thus $(K \circ K) = 0$, for a nonzero ideal $\theta(I) = K$, of R . Hence $0 = (K \circ K) = (RK \circ K) = R(K \circ K) - [R, K]K = -[R, K]K = -[R, K]RK$. Then $[R, K]R[R, K] = 0$. Which yields by semiprimeness of R that $[R, K] = 0$, that is, K is a nonzero central ideal which is contained in R .

Moreover, by Fact 2(b) R must be commutative when it is prime. In this case, by our hypothesis, we have $2F(ab) = 2\theta(ab)$ for all $a, b \in I$. If $\text{char}(R) = 2$, we are done. So we assume that $\text{char}(R) \neq 2$ and then we show that it leads a contradiction. Then we have $F(ab) = \theta(ab)$. Which gives that,

$$\theta(a)F(b) + \varphi(b)d(a) = \theta(a)\sigma(b) \quad (11)$$

for all $a, b \in I$. Substituting a with ca in (13) we get,

$$\theta(c)\theta(a)F(b) + \varphi(b)\{\theta(c)d(a) + \varphi(a)d(c)\} = \theta(c)\theta(a)\theta(b) \quad (12)$$

for all $a, b, c \in I$. Multiplying (13) from left by $\theta(c)$ and then subtracting from (14), we have $\varphi(b)\varphi(a)d(c) = 0$ for all $a, b, c \in I$, thus $\varphi(I)\varphi(I)d(I) = 0$. Since $\varphi(I)$ is a nonzero ideal of R , this gives $\varphi(I)d(I) = 0$, a contradiction.

Theorem 2.5.: Let us consider a semiprime ring R with a nonzero ideal J and θ, φ two epimorphisms of R . Consider a generalized left (θ, φ) derivation F of R associated with a left (θ, φ) -derivation d of R such that $\varphi(J)d(J) \neq 0$. If $F([a, b]) = \theta(aob)$ holds for all $a, b \in J$, then a nonzero central ideal contained in R .

Moreover, R must be commutative when R is prime (in this case either $\text{char}(R) = 2$ or $\theta(J) = 0$).

Proof: At the beginning we assume $F \neq 0$. Then by our hypothesis

$$F([a, b]) = \theta(aob) \quad (13)$$

for all $a, b \in J$. Now substitute ba for a in (15) and get

$$F(b[a, b]) = \theta(b)F([a, b]) + \varphi([a, b])d(b) = \theta(b)\theta(aob) \quad (14)$$

for all $a, b \in J$. For all $a, b \in J$ this gives by (15) that

$$\varphi([a, b])d(b) = 0 \tag{15}$$

which is identical with (7) in Theorem 2.3. By the same argument in Theorem 2.3, we conclude the result.

Next we consider $F=0$. Then we get $\theta(aob)=0$ holds for all $a, b \in J$. Arguing as before in Theorem 2.4 we get our required result.

Moreover, if R is prime, then by Fact-2(b), R must be commutative. in this case, by our hypothesis, we have $2\theta(J)\theta(J)=0$. This implies either $\theta(J)=0$ or $char(R)=2$.

Theorem 2.6. *Let us consider a semiprime ring R with two epimorphisms θ, φ and J a nonzero ideal of R . Assume that F is a generalized left (θ, φ) derivation of R associated with a left (θ, φ) derivation d of R such that $\varphi(J)d(J) \neq 0$. If $F(aob) = \theta([a, b])$ holds for all $a, b \in J$, then R contains a nonzero central ideal.*

Moreover, R must be commutative when R is prime.

Proof: First we consider that $F \neq 0$. Then by our hypothesis we have for all $a, b \in J$

$$F(aob) = \theta([a, b]) \tag{16}$$

Substituting ba for a in (18), we get

$$F(b(aob)) = \theta(b)F(aob) + \varphi(aob)d(b) = \theta(b)\theta([a, b]), \tag{17}$$

which gives by (18)

$$\varphi(aob)d(b) = 0 \tag{18}$$

for all $a, b \in I$. This is identical with (11) in Theorem 2.4, and so arguing as before in Theorem 2.4 we find our required results. Next $F=0$ case is same as that of Theorem 2.4 and then it is obvious to get our conclusion. The case of prime ring is similar as above.

Theorem 2.7. *Let us consider a semiprime ring R with two epimorphisms θ, φ of R and a nonzero ideal I of R . Consider a generalized left (θ, φ) derivation $F: R \rightarrow R$ associated with a left (θ, φ) -derivation $d: R \rightarrow R$ such that $\varphi(I)d(I) \neq 0$. If $F(ab) \pm \theta(ab) \in Z(R)$ holds for all $a, b \in I$, then a nonzero central ideal contained in R .*

Proof. First we consider that $F \neq 0$. Then by our hypothesis, for all $a, b \in I$, we have

$$F(ab) \pm \sigma(ab) \in Z(R) \tag{19}$$

Puttig ra for a , where $r \in R$, it gives

$$\begin{aligned} F(rab) \pm \theta(rab) &= \theta(r)F(ab) + \varphi(ab)d(r) \pm \theta(r)\sigma(ab) \\ &= \theta(r)(F(ab) \pm \theta(ab)) + \varphi(ab)d(r) \in Z(R) \end{aligned} \tag{20}$$

for all $a, b \in I$ and $r \in R$. Commuting both sides of (22) with $\theta(r)$ and then using (21), we have

$$[\varphi(ab)d(r), \theta(r)] = 0 \tag{21}$$

for all $a, b \in I$ and $r \in R$. Now we put $a = ta$ in (23), for $t \in R$, and get

$$[\varphi(t), \theta(r)]\varphi(ab)d(r) = 0 \tag{22}$$

for all $a, b \in I$ and for $r, t \in R$. That is $[R, \theta(r)]\varphi(I)\varphi(I)d(r) = 0$ for all $r \in R$. As $\varphi(I)$ is a nonzero ideal of R ,

$$[R, \theta(r)]R\varphi(I)\varphi(I)d(r) = 0 \tag{23}$$

for all $r \in R$. By using semiprimeness of R , R contain a family $P = \{P_i / i \in \mathbb{A}\}$ of prime ideals such that $\bigcap P_i = \{0\}$. For a typical member P_i of P and $a \in I$, and $r \in R$, (25) shows that $[R, \theta(r)] \subseteq P$ or $\varphi(I)\varphi(I)d(r) \subseteq P$. For fixed P_i , these two conditions hold for which set of $r \in R$ are formed additive subgroups of R whose union is R ; that is, $[R, \theta(r)] \subseteq P_i$ or $\varphi(I)\varphi(I)d(r) \subseteq P_i$ that is $[R, R] \subseteq P_i$ or $\varphi(I)\varphi(I)d(R) \subseteq P_i$. Together of these two conditions imply that $[R, \varphi(I)]\varphi(I)d(R) \subseteq P_i$ for any $P_i \in P$. Therefore, $[R, \varphi(I)]\varphi(I)d(R) \subseteq \bigcap_{\alpha \in \mathbb{A}} P_\alpha = 0$, that is $[R, \varphi(I)]\varphi(I)d(R) = 0$. In particular, $[R, \varphi(I)]\varphi(I)d(I) = 0$. Hence we obtain our conclusion by using Lemma 2.1.

Next, we take $F=0$. Then we get $\theta(ab) \in Z(R)$ for all $a, b \in I$, that is $\theta(I)^2 \subseteq Z(R)$. Also $\theta(I)^2$ is an ideal of R and it is nonzero, hence we obtain our conclusion.

Corollary 2.8. *Let us consider a prime ring R with a nonzero ideal I and two epimorphisms θ and φ of it such that $\theta(I) \neq 0$ and $\varphi(I) \neq 0$. Consider a generalized left (θ, φ) derivation $F: R \rightarrow R$ associated with a left (θ, φ) derivation $d: R \rightarrow R$. Suppose $F(a_1a_2) \pm \theta(a_1a_2) \in Z(R)$ holds for all $a_1, a_2 \in I$. Then one of the following two holds: (1) R is commutative; (2) $F(a_1) = \tau\theta(a_1) + \zeta(a_1)$ for all $a_1 \in I$, for an additive Right θ -multiplier mapping $\zeta: I \rightarrow Z(R)$ is.*

Proof. By Theorem 2.7, either R is commutative or $d(I) = 0$. We obtain our conclusion (1) if R is commutative. Now we suppose that $d(I) = 0$. By Fact-4, $d(R) = 0$ and hence $F(a_1a_2) = \theta(a_1)F(a_2)$, this implies that, F is a right θ -multiplier map. Hence by our hypothesis, we have $\theta(a_1)F(a_2) \pm \theta(a_1)\theta(a_2) \in Z(R)$ for all $a_1, a_2 \in I$. This yield

$$\theta(a_1)(F(a_2) \pm \theta(a_2)) \in Z(R) \tag{24}$$

for all $a_1, a_2 \in I$. Commuting both sides of (26) with $F(a_2) \pm \theta(a_2)$ we get

$$[\theta(a_1), F(a_2) \pm \theta(a_2)](F(a_2) \pm \theta(a_2)) \in Z(R) \tag{25}$$

for all $a_1, a_2 \in I$. Interchanging a_1 with ta_1 , for $t \in R$, we get

$$\begin{aligned} 0 &= [\theta(t)\theta(a_1), F(a_2) \pm \theta(a_2)](F(a_2) \pm \theta(a_2)) \\ &= [\theta(t)[\theta(a_1), F(a_2) \pm \theta(a_2)](F(a_2) \pm \theta(a_2)) + [\theta(t), F(a_2) \pm \theta(a_2)]\theta(a_1)(F(a_2) \pm \theta(a_2)) \\ &= [\theta(t), F(a_2) \pm \theta(a_2)]\theta(a_1)(F(a_2) \pm \theta(a_2)) \end{aligned} \tag{28}$$

for all $a_1, a_2 \in I$ and $t \in R$. By the primeness of R , for each $a_2 \in I$, either $F(a_2) \pm \theta(a_2) = 0$ or $[\theta(t), F(t) \pm \theta(a_2)] = 0$. Both cases implies that $[\theta(t), F(a_2) \pm \theta(a_2)] = 0$ for all $a_2 \in I$ and $t \in R$. It yields that $F(a_1) \pm \theta(a_1) = \zeta(a_1) \in Z(R)$ for all $a_1 \in I$, that is for all $a_1 \in I$, $F(a_1) = \mp \theta(a_1) + \zeta(a_1)$ where $\zeta : I \rightarrow Z(R)$ is an additive map. Since F is a right θ -multiplier map, ζ is also a right θ -multiplier map, which is our conclusion (2).

Theorem 2.9. Let us consider a semiprime ring R with nonzero ideal I and an epimorphism $\theta: R \rightarrow R$. Let us take a generalized left (θ, θ) derivation $F: R \rightarrow R$ associated with a left (θ, θ) -derivation $d: R \rightarrow R$ with the condition $\theta(I)d(I) \neq 0$. If $F(ab) \pm \theta(ba) \in Z(R)$ holds for all $a, b \in I$, then a nonzero central ideal contained in R .

Proof. First we consider that $F \neq 0$. Let us consider the case

$$F(ab) - \theta(ba) \in Z(R) \quad (29)$$

for all $a, b \in I$. Now substituting ca for a in (29), where $c \in I$, we get,

$$[[\theta(b), \theta(c)]\theta(a), \theta(c)] - [\theta(ab)d(c), \theta(c)] = 0 \quad (26)$$

for all $a, b, c \in I$. Commuting both sides of (30) with $\theta(c)$ and then using (29), we obtain

$$[[\theta(b), \theta(c)]\theta(a), \theta(c)] - [\theta(ab)d(c), \theta(c)] = 0 \quad (27)$$

for all $a, b, c \in I$. Replacing a with ac in (31), we get

$$[[\theta(b), \theta(c)]\theta(a), \theta(c)]\theta(c) + [\theta(acb)d(c), \theta(c)] = 0 \quad (28)$$

for all $a, b, c \in I$. Multiplying (31) from right by $\theta(c)$ and subtracting from (32), we obtain, for all $a, b, c \in I$, that

$$[\theta(a)[\theta(b)d(c), \theta(c)], \theta(c)] = 0 \quad (29)$$

We can replace $\theta(b)$ by $d(c)\theta(b)$, as $\theta(I)$ is a nonzero ideal of R , in (33) and then we get

$$[\theta(a)[d(c)\theta(b)d(c), \theta(c)], \theta(c)] = 0 \quad (30)$$

for all $a, b, c \in I$. As $\theta(I)$ is an ideal of R , we can write $d(c)\theta(b)d(c)\theta(a)$ in place of $\theta(a)$ in (34), and then we have $[d(c)\theta(b)d(c)\theta(a)[d(c)\theta(b)d(c), \theta(c)], \theta(c)] = 0$, and hence

$$d(c)\theta(b)d(c)[\theta(a)[d(c)\theta(b)d(c), \theta(c)], \theta(c)] + [d(c)\theta(b)d(c), \theta(c)]\theta(a)[d(c)\theta(b)d(c), \theta(c)] = 0. \quad (31)$$

By using (34), it yields

$$[d(c)\theta(b)d(c), \theta(c)]\theta(a)[d(c)\theta(b)d(c), \theta(c)] = 0 \quad (32)$$

for all $a, b, c \in I$. Replacing a by ta , for $t \in R$, and left multiplying $\theta(a)$, we get

$$\theta(a)[d(c)\theta(b)d(c), \theta(c)]\theta(t)\theta(a)[d(c)\theta(b)d(c), \theta(c)] = 0 \quad (33)$$

for all $a, b, c \in I$ and $t \in R$. The semiprimeness of R gives

$$\theta(a)[d(c)\theta(b)d(c), \theta(c)] = 0 \quad (34)$$

for all $a, b, c \in I$. That is, for all $a, b, c \in I$

$$\theta(a)(d(c)\theta(b)d(c)\theta(c) - \theta(c)d(c)\sigma(b)d(c)) = 0 \quad (35)$$

Now putting $\theta(b) = \theta(b)d(c)\theta(w)$ for $b, c, w \in I$, above relation gives

$$\theta(a)(d(c)\theta(b)d(c)\theta(w)d(c)\theta(c) - \theta(c)d(c)\theta(b)d(c)\theta(w)d(c)) = 0. \quad (36)$$

Using (39), this gives

$$\theta(a)(d(c)\theta(b)\theta(c)d(c)\theta(w)d(c) - d(c)\theta(b)d(c)\theta(c)\theta(w)d(c)) = 0, \quad (37)$$

which implies

$$\theta(a)d(c)\theta(b)[d(c), \theta(c)]\theta(w)d(c) = 0 \quad (38)$$

for all $a, b, c, w \in I$. From this it is obvious that

$$\theta(a)[d(c), \theta(c)]\theta(b)[d(c), \theta(c)]\theta(w)[d(c), \theta(c)] = 0 \quad (39)$$

for all $a, b, c, w \in I$, thus $(\theta(I)[d(c), \theta(c)])^3 = 0$ for all $c \in I$. As a semiprime ring does not contain nonzero nilpotent ideal it follows that $\theta(I)[d(c), \theta(c)] = 0$, and hence $\theta(a)[d(c), \theta(c)] = 0$ for all $a, c \in I$. As $\theta(I)$ is an ideal of R , replacing $\theta(a)$ by $\theta(a)d(as)$ and $d(a)\theta(as)$, for $s \in R$, we get $\theta(a)d(as)[d(c), \theta(c)] = 0$ and $d(a)\theta(as)[d(c), \theta(c)] = 0$ respectively. Subtracting these two one from another, we get for all $a, c \in I$ and $s \in R$ that $[d(a), \theta(a)]s[d(c), \theta(c)] = 0$. Particularly, for all $a \in I$ and $s \in R$,

$$[d(a), \theta(a)]s[d(a), \sigma(a)] \quad (40)$$

that is for all $a \in I$, $[d(a), \theta(a)]R[d(a), \theta(a)] = 0$. Hence the semiprimeness of R gives $[d(a), \theta(a)] = 0$ for all $a \in I$. Then using Lemma 2.2 we arrived at our desired goal.

Next we consider $F=0$. Then we get $\theta(ba) \in Z(R)$ holds for all $a, b \in I$, that is $\theta(I)^2$ is central. Now $\theta(I)^2$ is an ideal of R which is also nonzero. Thus we obtain that a nonzero central ideal contained in R . Proceeding in the same way by considering the case $F(ab) + \theta(ba) \in Z(R)$ for all $a, b \in I$ we can prove the same result.

Corollary 2.10. Let us suppose that R be a prime ring with a nonzero ideal I and an epimorphism θ , such that $\theta(I) \neq 0$. Consider a generalized left (θ, θ) derivation F of R associated with a left (θ, θ) derivation δ of R . If $F(uv) \pm \theta(vu) \in Z(R)$ happens for all $u, v \in I$, then R must be commutative.

Proof. Using Theorem 2.9 we get either R is commutative or $\delta(I) = 0$. If R is commutative, we are done. So we consider $\delta(I) = 0$. Then by Fact-4, $\delta(R) = 0$ thus $F(uv) = \theta(u)F(v)$, that is, F is a right θ -multiplier map. Hence by our hypothesis, we find

$$\theta(u)F(v) \pm \theta(v)F(u) \in Z(R) \quad (41)$$

for all $u, v \in I$. Changing u with wu in the above relation, we obtain

$$\theta(w)\theta(u)F(v) \pm \theta(v)\theta(w)\theta(u) \in Z(R) \quad (42)$$

for all $u, v, w \in I$. Multiplying (45) from left by $\theta(w)$ then subtracting from (46), we obtain

$$\theta(w)(\theta(u)F(v) - \theta(v)\theta(u)) \pm [\theta(v), \theta(w)]\theta(u) \in Z(R) \quad (43)$$

for all $u, v, w \in I$. Commuting both sides of (47) with $\theta(w)$ and then using (45), we have

$$[[\theta(v), \theta(w)]\theta(u), \theta(w)] = 0 \quad (44)$$

for all $u, v, w \in I$. Writing ut in place of u , for $t \in I$, we get

$$\begin{aligned} 0 &= [[\theta(v), \theta(w)]\theta(u)\theta(t), \theta(w)] \\ &= [[\theta(v), \theta(w)]\theta(u), \theta(w)]\theta(t) + [\theta(v), \theta(w)]\theta(u)[\theta(t), \theta(w)] \quad (45) \\ &= [\theta(v), \theta(w)]\theta(u)[\theta(t), \theta(w)] \end{aligned}$$

for all $u, v, w, t \in I$. Now $\theta(I)$ is a nonzero ideal of R and using the primeness of R we have from the above relation that $[\theta(I), \theta(I)] = 0$. Then by Fact-2, we conclude that R must be commutative.

Example: Example 3.1.

Let us choose $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} / a, b, c \in \square \right\}$, Then it is

immediate that $I = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} / b, c \in \square \right\}$ is a nonzero ideal

of R . Now for $x = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in I$ and $r = \begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix} \in$

R , we see that $[r, x] \neq 0$. Thus I is noncentral. Obviously, R is not semiprime, since $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$ Let us

define mappings, $F, d, \theta, \varphi: R \rightarrow R$, by

$$F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \theta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}, \varphi \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \\ \tau \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \text{ for all } a, b, c \in \mathbb{Z}. \end{aligned}$$

It is easy to see that θ, φ are epimorphisms of R and $F(ab) = \theta(a)F(b) + \varphi(b)d(a)$ holds for all $a, b \in R$ that is F is a generalized left (θ, φ) -derivation of R and d is a left (θ, φ) -derivation of R . Now $F(ab) \pm \theta(ab) \in Z(R)$ for all $a, b \in I$, but I is noncentral. Hence we conclude that the hypothesis of semiprimeness in Theorem 2.7 is essential.

Conclusion

In this article I studied the relationship between the behavior of left generalized (θ, φ) -derivations on a semiprime ring R (and also studied the same cases in prime ring) and the structure of the ring R . In all the identities, studied here, it has found that if we consider the ring R as semiprime then R always contains a central ideal and if R is taken as a prime ring then it becomes a commutative ring.

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