

Research Journal of Mathematical and Statistical Sciences \_\_\_\_\_\_ Vol. 3(5), 1-4, May (2015)

# Analyticity and Inversion Theorem for A New Version of Banach Space Valued Potential Transform

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Available online at: www.isca.in, www.isca.me

Received 29<sup>th</sup> March 2015, revised 14<sup>th</sup> April 2015, accepted 10<sup>th</sup> May 2015

#### Abstract

In this paper we have defined new versions of generalized Potential transform on Banach space. Then for new version of Banach space valued Potential transform, the analyticity and inversion theorem has been derived.

Keywords: Testing function spaces, Generalized function, Banach space, Potential transforms.

#### Introduction

The Theory of integral transforms has vast applications in solving differential equations, in various physical situations. As explained by Zemanian A.H.<sup>1</sup>, many times the functions in some situations take values from Banach space instead of Euclidean space. Hence some conventional integrals transforms are studied in the Banach space domain e.g. Tekale<sup>2</sup> discussed Banach spaced valued Stieltjes transform. Gudadhe<sup>3</sup> studied Banach spaced valued Mellin transform. Koh, Li<sup>4</sup> discussed on Banach spaced valued Hankel transform. Recently in 2012 Holmes<sup>5</sup> had extended Gaussians Radon transform on Banach space.

We have already extended Potential transform<sup>6</sup> to the Banach space valued generalized function and proved its inversion theorem.

The objective of this paper is to study a new version of generalized Potential transform in Banach space. Hence in section 2, we have defined Banach space valued new version of Potential transform by stating suitable testing function space. Section 3 is devoted to discuss analyticity of new integral transform and 4th section proves inversion theorem of Banach space valued new version of Potential transform.

# Definition

We first define testing function space  $P_{2n, c,d,\alpha}$ , new version of Potential transform.

 $P_{2n. c.d.\alpha}(A)$  denotes the space of all complex valued smooth

functions  $\psi(t)$  on which functional  $i_{c,d,k}$  defined by

$$P_{2n,c,d,\alpha}(A) = \{ \psi : \psi \in E_{+}(A); i_{c,d,k}(\psi) = \sup_{0 < t < \infty} \left\| \lambda_{c,d}(t)(tD_{t})^{k} \{ t\psi(t) \} \right\|_{A} \le C_{k} L^{k} k^{k\alpha}, k = 0, 1, 2, 3, .... \}$$

The constant  $C_k$  and L depend on function and

$$\lambda_{{}_{c,d}}(t) \!=\! \begin{cases} t^{-c}\,, & 0 \!<\! t \!<\! 1 \\ t^{-d}\,, & 1 \!\leq\! t \!<\! \infty \end{cases}$$

where c and d are real numbers.

It can be easily proved that  $\frac{t^{2n-1}}{t^{2n}+x^{2n}} \in P_{2n, c,d,\alpha}(A)$ . Hence we define a new version of Banach space valued Potential transform as

Then the Potential transform of a regular function, f(t) is defined as,

Let  $f \in [D_+; A]$  is Banach space valued Potential transformable function, if there exists two members  $\sigma_1$ ,  $\sigma_2 \in [-\infty, \infty]$ , such that  $\sigma_1 < \sigma_2$ ,  $f \in [P_{2n}(\sigma_1, \sigma_2); A]$  and in addition  $f \notin [P_{2n}(w, z); A]$  if either  $w < \sigma_1$  or  $z > \sigma_2$ , where  $[P_{2n}(w, z); A]$  is as defined in 3.1.3, and  $\Omega_f = \{x : \sigma_1 < \operatorname{Re}(x) < \sigma_2\}, \frac{t^{2n-1}}{x^{2n} + t^{2n}} \in P_{2n}(\sigma_1, \sigma_2).$ 

Potential transform of a regular function f(t), is defined as,

$$P_{2n} \{f(t); x\} = P_{2n}(x) = \left\langle f(t), \frac{t^{2n-1}}{t^{2n} + x^{2n}} \right\rangle,$$

Now we shall show that Banach space valued  $P_{2n}(x)$  is analytic.

Analyticity Theorem for a New Version of Banach Space Valued Potential Transform  $\begin{array}{ll} \textbf{Theorem:} \ If \ P_{2n}\left\{f\left(t\right); x \ \right\} = P_{2n}\left(x\right) \ as \ defined \ in \ (2) \ for \ x \in \Omega_{f} \ , \\ here \quad \Omega_{f} = \left\{x: \sigma_{1} < \text{Re}\left(x\right) < \sigma_{2}\right\} \quad also \ P_{2n} \in \left[D\left(A\right), B\right] \quad then \\ P_{2n}\left(x\right) \ is \ a \ \left[A, B\right] \ valued \ analytic \ function, \ here \ x \in \Omega_{f} \ . \ For \\ nonnegative \quad integer \ q \ , \ where \quad P_{2n}^{\quad (q)}\left(x\right) = \left\langle f\left(t\right), \ K^{q}\left(x, t\right) \right\rangle, \end{array}$ 

$$K(x, t) = \frac{t^{2n-1}}{t^{2n} + x^{2n}}, \text{ for } x \in \Omega_{f}.$$

**Proof:** By induction method, we prove the theorem on q. Let X be an arbitrary but fixed point in  $\Omega_f$ . Here  $P_{2x}(x) = \langle f(t), K(x, t) \rangle$ , has meaning, because  $f \in [P(\sigma_1, \sigma_2); A]$  and  $\frac{t^{2n-1}}{t^{2n} + x^{2n}} = K(x, t) \in P(\sigma_1, \sigma_2)$ .

The real positive numbers, we choose a,b,r and  $r_1$  such that  $\sigma_1 < c < \text{Re}(x-r) < \text{Re}(x-r_1) < d < \sigma_2$  also  $\Delta x$  be the complex increment such that  $|\Delta x| < r$ .

$$\frac{P_{2n}(x + \Delta x) - P_{2n}(x)}{\Delta x} - \left\langle f(t), \frac{\partial}{\partial x} \frac{t^{2n-1}}{t^{2n} + x^{2n}} \right\rangle = \left\langle f(t), \psi_{2n,\Delta x}(t) \right\rangle,$$

Therefore,

$$\begin{split} &\frac{1}{\Delta x} \left\{ \! \left\langle f(t), \frac{t^{2n-l}}{(x+\Delta x)^{2n} + t^{2n}} \right\rangle \! - \! \left\langle f(t), \frac{t^{2n-l}}{x^{2n} + t^{2n}} \right\rangle \! \right\} \! - \! \left\langle f(t), \frac{\partial}{\partial x} \frac{t^{2n-l}}{x^{2n} + t^{2n}} \right\rangle \! = \! \left\langle f(t), \psi_{2n,\Delta x}(t) \right\rangle \\ &\Rightarrow \left\langle f(t), \left\{ \frac{1}{\Delta x} \! \left( \frac{t^{2n-l}}{(x+\Delta x)^{2n} + t^{2n}} \! - \! \frac{t^{2n-l}}{x^{2n} + t^{2n}} \right) \! - \! \frac{\partial}{\partial x} \frac{t^{2n-l}}{x^{2n} + t^{2n}} \right\} \! \right\} \! = \! \left\langle f(t), \psi_{2n,\Delta x}(t) \right\rangle \end{split}$$

where

$$\psi_{2n,\Delta x}(t) = \frac{1}{\Delta x} \left( \frac{t^{2n-1}}{(x+\Delta x)^{2n} + t^{2n}} - \frac{t^{2n-1}}{x^{2n} + t^{2n}} \right) - \frac{\partial}{\partial x} \frac{t^{2n-1}}{x^{2n} + t^{2n}}$$

which can be write down as follows,

$$\begin{split} \psi_{2n,\Delta x}(t) &= \frac{1}{\Delta x} \left[ K(x + \Delta x, t) - K(x, t) \right] - \frac{\partial}{\partial x} K(x, t), \\ D^{q} \left\{ \psi_{2n,\Delta x}(t) \right\} &= \frac{1}{\Delta x} \left[ K^{q}(x + \Delta x, t) - K^{q}(x, t) \right] - \frac{\partial}{\partial x} K^{q}(x, t), \end{split}$$

where,  $K^{q}(x, t) = \frac{\partial^{q}}{\partial x^{q}} K(x, t)$ .

To proceed, let C be the circle with centre x and radius  $r_1$ . Let us restrict  $r_1$  such that C lies entirely with  $\Omega_f$  and  $0 < r < r_1$ .

Using Cauchy's integral formula we get,

$$D_{t}^{q}\left\{\psi_{2n,\Delta x}(t)\right\} = \frac{D_{t}^{q}}{2\pi i\Delta x} \int_{C} \left[\frac{1}{\left[\xi - (x + \Delta x)\right]} - \frac{1}{\left[\xi - x\right]}\right] K(\xi, t) \, d\xi - \frac{1}{2\pi i} \int_{C} \frac{K(\xi, t)}{(\xi - x)^{2}} \, d\xi,$$

 $\xi \in C$  therefore,

$$D_t^q \left\{ \psi_{2n,\Delta x}(t) \right\} = \frac{\Delta x}{2\pi i} \int_C \frac{D_t^q K(\xi,t)}{\left[\xi - (x + \Delta x)\right] (\xi - x)^2} d\xi$$

Now for fixed  $\xi \in C$ ,  $0 < t < \infty$   $D_t^q K(\xi, t)$  is a continuous function on a compact subset of  $\Omega_f$ , hence it is bounded. Therefore  $|D_t^q K(\xi, t)| \le N$ 

Moreover, 
$$|\xi - x - \Delta x| > r_1 - r > 0$$
 and  $|\xi - x| = r_1$   
Therefore  $\|\lambda_{c,d}(t) D_t^q K(\xi,t)\|_A \le \frac{|\Delta x|}{2\pi i} \int_C \frac{N}{(r - r_1)r_1^2} d\xi$   
Therefore  $\|\lambda_{c,d}(t) D_t^q K(\xi,t)\|_A \le \frac{|\Delta x|N}{(r - r_1)r_1^2}$  for some

constant N.

Now 
$$\frac{|\Delta x| N}{(r-r_1){r_1}^2} \rightarrow 0$$
 as  $|\Delta x| \rightarrow 0$ 

This shows that  $\psi_{2n,\Delta x}(t)$  converges to zero in  $P_{2n,c,d}(A)$  and hence the proof is complete.

# **Inversion Theorem for** $P_{2n}(x)$

For the inversion theorem for new version of Banach space valued Potential transform we have to use another version of Laplace transform on Banach space. Hence first we defined Banach space valued second version of Laplace transform

#### **4.1 Banach Space Valued Second Version Laplace Transform:** Given any $a, b \in R$ , set

Let 
$$\lambda_{a,b} = \begin{cases} e^{at^2} & 0 \le t < \infty \\ e^{bt^2} & -\infty < t < 0 \end{cases}$$

 $\begin{array}{ll} L_{2,a,b}(A) \text{ is the linear spaces of all } A \text{ valued smooth} \\ \text{functions } & \text{on } R \text{ Such that} \\ L_{2,a,b}(A) = \left\{ \psi : \psi \in E_{+}(A); i_{c,d,k}(\psi) = \sup_{0 < t < \infty} \left\| \lambda_{a,b}(t) \psi^{k}(t) \right\|_{A} < \infty \right\}$ 

 $L_{2,a,b}(A)$  is complete and therefore Frechet space under topology generated by the multinorms  $\left\{ \begin{array}{l} \gamma_{a,b,k} \end{array} \right\}_{k=0}^{\infty}$ .

**International Science Congress Association** 

\_\_\_ISSN 2320-6047

Res. J. Mathematical and Statistical Sci.

f is said to be Banach space valued second version Laplace transformable, if there exists two elements  $\sigma_1$  and  $\sigma_2$  in the extended real line [−∞,∞] such  $\sigma_1 < \sigma_2$ , that  $f \in [L_2(\sigma_1, \sigma_2); A]$  and in addition  $f \notin [L_2(w, z); A]$  if either  $w < \sigma_1$  or  $z > \sigma_2$  with the open strip  $\Omega_{\rm f} = \{ {\rm x} : \sigma_1 < {\rm Re} \, {\rm x} < \sigma_2 \}, \text{ will be called the strip definition for}$ the  $L_2$  transform of f. By the aforementioned identification,  $f \in [L_2(\sigma_1, \sigma_2); [A; B]]$  also. Thus, we may define the  $L_2$ transform of f as a mapping of  $\Omega_{f}$  into [A;B] by  $te^{-t^2x^2} \in L_2[(\sigma_1, \sigma_2); A]$ , then the second version Laplace transform is defined as,

$$F(x) = L_2 \{ f(t); x \} = \langle f(t); te^{-t^2 x^2} \rangle.$$
  
L\_2(x) is a Avalued function and also L\_2(x) is analytic.

So the second version of Laplace transform i.e.  $L_2$  transform is given by,  $L_2 \{ f(t); x \} = F(x) = \int_{0}^{\infty} t f(t) \exp(-t^2 x^2) dt$ 

Also if  $te^{-t^2x^2} \in L_2(w,z,A)$ , so that the second version of Laplace transform i.e.  $L_2$  transform of [A,B] valued distribution is defined as  $L_2^{AB}(x) = \langle f(t); te^{-t^2x^2} \rangle$  and here LHS is [A,B] valued distribution. Similarly if  $te^{-t^2x^2} \in L_2(\sigma_1,\sigma_2)$  and  $f \in [L_2(\sigma_1,\sigma_2); A]$  then we can define A- valued  $L_2$  transform denoted by  $L_2^A(x)$ .

4.2 Relations between Generalized and Classical Transforms of Laplace and Potential: There is no need of extending these transforms to Banach space, as Laplace transform is already extended to Banach space valued generalized function  $in^8$ .

Generalized  $L_{2n}$  transform can be expressed as,

$$L_{2n}{f(t); x} = \langle f(t); t^{2n-1} \exp(-t^{2n}x^{2n}) \rangle$$

 $L_{2n}$  transform is related to the classical Laplace transform and the second version of the generalized Laplace transform i.e.  $L_2$  transform by the following relationship,

$$L_{2n} \{ f(t); x \} = \frac{1}{2n} L \left\{ f\left(t^{\frac{1}{2n}}\right); x^{2n} \right\}$$
$$L_{2n} \{ f(t); x \} = \frac{1}{n} L_2 \left\{ f\left(t^{\frac{1}{n}}\right); x^n \right\}.$$

**4.3 Lemma:** The identity  $P_{2n}^{A,B}\{f(t); x\} = 2n L_{2n}^{A,B}\{L_{2n}^{A}\{f(t); y\}; x\}$  hold true, provided that the integrals involved converges absolutely, in the respective Banach spaces.

**Proof**: Consider  $L_{2n}^{A,B} \{ L_{2n}^{A} \{ f(t); y \}; x \}$ 

By definition of 
$$L_{2n}$$
 transform, we have,  
 $L_{2n}^{A,B} \left\{ L_{2n}^{A} \left\{ f(t); y \right\}; x \right\} = \left\langle L_{2n}^{A} [f(t); y]; y^{2n-1} e^{-x^{2n}y^{2n}} \right\rangle$ By  
changing the order of the integration, we have,  
 $L_{2n}^{A,B} \left\{ L_{2n}^{A} \left\{ f(t); y \right\}; x \right\} = \left\langle f(t) t^{2n-1}; \left\langle y^{2n-1} \exp\{-x^{2n}y^{2n}\right) \exp\{-t^{2n}y^{2n}\right) \right\rangle$   
 $= \left\langle f(t) t^{2n-1}; \left\langle y^{2n-1}; \exp\{-y^{2n}(x^{2n}+t^{2n})\} \right\rangle$ 

Set 
$$y^{2n} = m \implies y = m^{\frac{1}{2n}}$$
  
 $L_{2n}^{A,B} \left\{ L_{2n}^{A} \left\{ f(t); y \right\}; x \right\} = \frac{1}{2} \left\langle f(t) ; \frac{t^{2n-1}}{(t^{2n} + x^{2n})} \right\rangle$   
 $\implies P_{2n}^{A,B} \left\{ f(t); x \right\} = 2n L_{2n}^{A,B} \left\{ L_{2n}^{A} \left\{ f(t); y \right\}; x \right\}$ 

4.4. Inversion Theorem: Let  $f \in [D(A); B]$  and  $P_{2n}{f(t); x} = P_{2n}(x) = \langle f(t); \frac{t^{2n-1}}{x^{2n} + t^{2n}} \rangle$  for  $x \in \Omega_{f}$ ,

 $\Omega_{f} = \{ x : \sigma_{1} < \operatorname{Re}(x) < \sigma_{2} \}, \quad \frac{t^{2n-1}}{x^{2n} + t^{2n}} \in P(\sigma_{1}, \sigma_{2}), \quad \text{then}$ convergence in [D(A); B],

$$f(t) = \frac{1}{2n\pi i} \left\langle \frac{1}{\pi i} \left\langle P_{2n}\left(x^{\frac{1}{2n}}\right); e^{xy^{2n}} \right\rangle; e^{yt^{2n}} \right\rangle.$$

**Proof:** Here we want to show that  $f(t) = \frac{1}{2n\pi i} \left\langle \frac{1}{\pi i} \left\langle P_{2n}\left(x^{\frac{1}{2n}}\right); e^{xy^{2n}} \right\rangle; e^{yt^{2n}} \right\rangle.$ 

By lemma 4.3, omitting the superscripts, we get

$$P_{2n} \{ f(t); x \} = 2n L_{2n} \{ L_{2n} \{ f(t); y \}; x \}$$
  
that is  $\frac{1}{2n} P_{2n}(x) = L_{2n} \{ L_{2n} \{ f(t); y \}; x \}$ 

The required result is the natural consequence of above lemma 4.3 and lemmas 7.2.1, and 7.2.2 in [4] i.e. in our research paper "Banach Space Valued Potential Transform".

# Conclusion

The generalized integral transform proved to be an important tool in system analysis. We observed the importance and a special requirement of Banach space valued generalized function theory. So that we have studied Analyticity theorem and Inversion theorem of Banach space valued Potential transform having applications in number of fields, whenever the functions involved in that are Dirac delta type singular functions.

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