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On a Right Truncated Generalized Gaussian distribution

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Abstract

Generalized Gaussian distribution is useful in analyzing several data sets arising at places at image processing, speech recognition, signal processing, statistical quality control, agricultural experimentation, industrial experimentation and biological experiments. In this paper, a right truncated generalized Gaussian distribution is introduced. The various distribution properties such as the distribution function, moments, skewness, kurtosis, hazard function and survival function are derived. The distribution of the r^{th} order statistics and the median distribution are also derived. Some inferential aspects of the distribution are also studied.

Keywords: Right truncated distribution, generalized Gaussian distribution, distributional properties, order statistics.

Introduction

Generalized Gaussian distribution has been proposed for modeling atmospheric noise, subband encoding of audio-video signals, impulse noise, blind signals separation etc. Varanasi M.K et al[1], Choi, S. Cichocki et al[2], Wu et al[3], Armando et al[4]. Edgeworth[5] has considered the possibility of polynomial transformations to normality. Kameda T[6] has pioneered the idea of probability plot to indicate the form of transformation. Johnson N.L[7] has introduced a system of frequency curves generated by method of translation analogy to pearsonian system of distributions using log-normal and or unit normal distribution. By choosing an initial distribution Gram-Charlier series distributions are generated by Edgeworth using the normal distribution. Plucinska[8] used generalized gamma distributions one for negative and one for positive values of arguments to construct a new class of distribution functions. She also developed in 1965 the distributions by reflecting the generalized gamma distribution about the origin. Borgi.O[9] has also considered similar reflection of the standard gamma distribution. The main difficulty in these distributions is that the density is zero in general at the point of symmetry. Srinivasa Rao et al[10] have generated a class of symmetric distributions using Laplace distribution. Anithakumari et al[11] developed and analyzed a left truncated generalized Gaussian distribution. There it is assumed that the variate on the study follows a generalized Gaussian distribution and constrained with a finite value on the left end. This distribution work well in some cases where there is a minimum threshold for the variate under study. However, in some other datasets arising at quality control, Agricultural experiments, reliability study, the variable under study is having constrained on the right end. i.e., there is an upper bound for the variable. For example, in man power modeling there is an upper bound for the complete length of service known as age of superannuation. For these sort of situations it is needed to consider right truncated generalized Gaussian distribution.

In this paper, we develop and analyze a right truncated generalized Gaussian distribution. The various distributional properties such as the probability density function, the distribution function, the four moments, the skewness, the kurtosis, the hazard function and survival function are derived. The order statistics of the distribution are also studied. Some inferential properties related to the parameters of the distribution are discussed. A numerical illustration is also presented.

Right Truncated Generalized Gaussian distribution

A Continuous random variable X is said to be a three parameter generalized Gaussian distribution if its probability density function (p.d.f) is of the form

$$f(x;\mu,\alpha,\beta) = \frac{\beta}{2\,\alpha\,\Gamma\left(\frac{1}{\beta}\right)} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \; ; \; -\infty < x < \infty \; ;$$

$-\infty < \mu < \infty \,; \alpha > 0 \,; \beta > 0$

Consider that the range variable is finite say $(-\infty, B)$. Then the probability density function (p.d.f) of a right truncated three parameter generalized Gaussian distribution is

$$f(x) = \frac{f(x;\mu,\alpha,\beta)}{F(B)}; \; -\infty < x < B; -\infty < \mu < B; \; \alpha > 0; \; \beta > 0$$

Where $$F(B)=\int_{-\infty}^{B}\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}dx \qquad (1)$$

The lower and upper truncation points are $-\infty$ *and* B respectively. Hence, the probability density function of three parameter right truncated generalized Gaussian distribution is

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \quad for \quad B<\mu \qquad (2)$$

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \quad for \quad B\geq\mu \qquad (3)$$

where, $\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)$ is an incomplete gamma function.

Distributional Properties

The various distributional properties of the right truncated generalized Gaussian distribution are discussed in this section

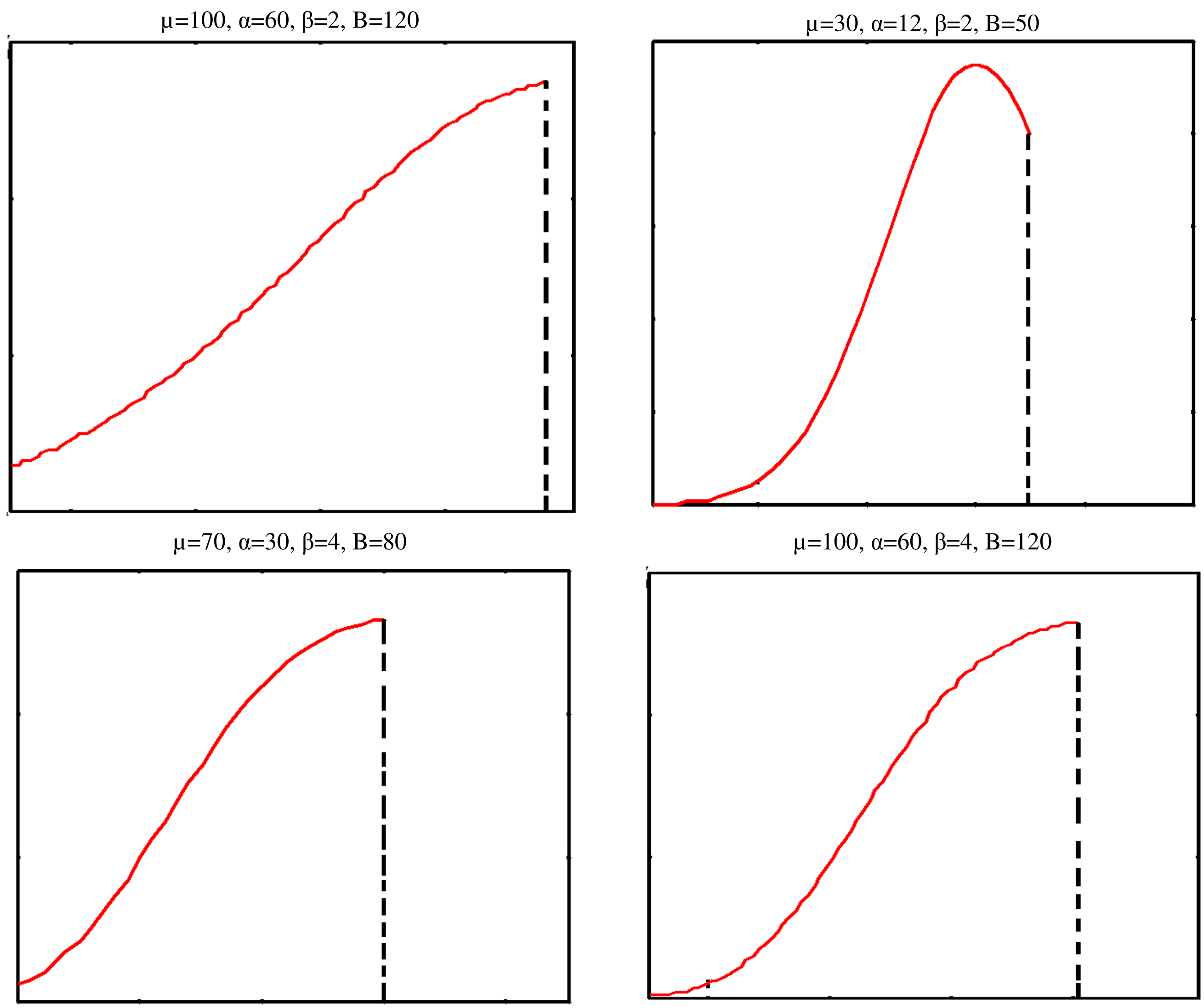

Figure-1
The frequency curves for different values of the right truncated generalized Gaussian distribution.

From figure1 it is observed that this distribution is uni-model distribution.

The distribution function of *X* is given by

$$F(x)=\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\quad for \quad B<\mu \tag{4}$$

$$F(x)=\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\quad for \quad B\geq\mu \tag{5}$$

where, $\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)$ is an incomplete gamma function.

The mean of the distribution is

$$E(X)=\mu+\alpha\left(\frac{\left(\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)\quad for \quad B<\mu \tag{6}$$

$$E(X)=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)\quad for \quad B\geq\mu \tag{7}$$

The median M of the distribution can be obtained by solving the equation (8) and (9)

$$\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\left(\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)\right)}=\frac{1}{2}\quad for \quad B<\mu \tag{8}$$

$$\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\left(\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)\right)}=\frac{1}{2}\quad for \quad B\geq\mu \tag{9}$$

The mode of the distribution can be obtained by solving the following equation (10) for x

$$f'(x)=\frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}=0 \tag{10}$$

This model is uni-model distribution

The raw moments of the distribution are

$$\mu'_r=\sum_{j=0}^{r}\binom{r}{j}\alpha^j\mu^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)-\gamma\left(\frac{j+1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)\quad for B<\mu \tag{11}$$

Similarly for $B\geq\mu$, the r[th] non central moment is

$$\mu'_r=\sum_{j=0}^{r}\binom{r}{j}\alpha^j\mu^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)\quad for\ B\geq\mu \tag{12}$$

The central moments of this distribution are

$$\mu_r=\sum_{j=0}^{r}\binom{r}{j}\alpha^j(-D)^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)-\gamma\left(\frac{j+1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)\quad for\ B<\mu \tag{13}$$

where $D=\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$

Similarly for $B\geq\mu$, the r[th] non central moment is

$$\mu_r=\sum_{j=0}^{r}\binom{r}{j}\alpha^j(-D)^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)\quad for B\geq\mu \tag{14}$$

where $D=\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$

(vii) The skewness of the distribution is

$$\beta_1 = \frac{\left(2S_1^3 - 3S_1S_2 + S_3\right)^2}{\left(S_2 - S_1^2\right)^3} \quad for \quad B < \mu$$

$$\beta_1 = \frac{\left(2P_1^3 - 3P_1P_2 + P_3\right)^2}{\left(P_2 - P_1^2\right)^3} \quad for \;\; B \ge \mu$$

Kurtosis of the distribution is

$$\beta_2 = \frac{3S_1^2\left(2S_2 - S_1^2\right) + S_4 - 4S_1S_3}{\left(S_2 - S_1^2\right)^2} \quad for \;\; B < \mu$$

$$\beta_2 = \frac{3P_1^2\left(2P_2 - P_1^2\right) + P_4 - 4P_1P_3}{\left(P_2 - P_1^2\right)^2} \quad for \; B \ge \mu$$

where $S_1 = \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right),$

$$S_3 = \left(\frac{\Gamma\left(\frac{4}{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right), \quad S_4 = \left(\frac{\Gamma\left(\frac{5}{\beta}\right) - \gamma\left(\frac{5}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)$$

and $P_1 = \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right), \quad P_4 = \left(\frac{\Gamma\left(\frac{5}{\beta}\right) + \gamma\left(\frac{5}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)$

The hazard rate function of the distribution is $h(x) = \frac{f(x)}{1 - F(x)}$

$$h(x) = \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{-\gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^\beta\right)} \quad for \;\; B < \mu \tag{15}$$

$$h(x) = \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^\beta\right)} \quad for \;\; B \ge \mu \tag{16}$$

The survival rate function $S(x)$ is $S(x) = 1 - F(x)$

$$S(x) = 1 - \frac{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)} \quad for \;\; B < \mu \tag{17}$$

$$S(x) = 1 - \frac{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)} \quad for \;\; B \ge \mu \tag{18}$$

Order Statistics of Right Truncated Three Parameter Generalized Gaussian distribution

The simple explicit form of the distribution function as given in equation (4) and (5) leads us to derive the order statistics connected with this right truncated three parameter generalized Gaussian distribution.

$$f(x) = \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)} \qquad B < \mu$$

$$f(x) = \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)} \qquad B \ge \mu \tag{19}$$

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics obtained from a random sample of size n from the generalized truncated Gaussian distribution having the probability density function of the form given in (19). The probability density function of s^th^ order statistics[12] is given by,

$$f_{s:n}(x) = D_{s:n}\left[F(x)\right]^{s-1}\left[1 - F(x)\right]^{n-s} f(x)$$

where $$D_{s:n} = \frac{n!}{(s-1)!(n-s)!} \tag{20}$$

Substituting f(x) and F(x) values given in this equation (19) and (5) in the equation (20), we get the probability density function of the s^{th} order statistics is given by

Case (i): For $B \ge \mu$

For $-\infty < x < 0$

$$f_{s:n}(x) = D_{s:n} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)} \sum_{q=0}^{s-1} \binom{s-1}{q} (-1)^q \left[\frac{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right]^{n+q-s}$$

For $0 < x < B$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{s+q-1} \quad (21)$$

Substituting f(x) and F(x) values given in this equation (19) and (4) in the equation (20), we get the probability density function of the s[th] order statistics is given by

Case (ii): For B < μ

For $-\infty < x < 0$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}$$

For $0 < x < B$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1} \quad (22)$$

The probability density function of the first order statistics is obtained by substituting $s=1$ in the equation (21)

Hence, case (i): For B ≥ μ,

For $-\infty < x < 0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1}$$

For $0 < x < B$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

The probability density function of the first order statistics is obtained by substituting $s=1$ in the equation (22)

Hence, case (ii): For B < μ,

For $-\infty < x < 0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1}$$

For $0 < x < B$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q} \quad (23)$$

The probability density function of the n[th] order statistics is obtained by substituting $s = n$ in equation (21)

Case (i): For B ≥ μ

For $-\infty < x < 0$

$$f_{n:n}(x) = \frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

For $0 < x < B$

$$f_{n:n}(x) = \frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1} \tag{24}$$

The probability density function of the n[th] order statistics is obtained by substituting $s = n$ in equation (22)

Case (ii): for B < μ

For $-\infty < x < 0$

$$f_{n:n}(x) = \frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q}$$

For $0 < x < B$

$$f_{n:n}(x) = \frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1} \tag{25}$$

Distribution of the Median: Let n be odd. The distribution of the median is obtained by substituting $s = \frac{n+1}{2}$ in equation (21) and equation (22).

For $-\infty < x < 0$

$$f_{M}(x) = \left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^{2}}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{\frac{n-1}{2}+q} \quad B \geq \mu$$

$$f_{M}(x) = \left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^{2}}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{\frac{n-1}{2}+q} \quad B < \mu$$

For $0 < x < B$

$$f_{M}(x) = \left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^{2}}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{\frac{n-1}{2}+q} \quad B \geq \mu \tag{26}$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q} \tag{27}$$

$$(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{x-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad B$$

Inferential Aspects of the Right Truncated Three Parameter Generalized Gaussian distribution

Method of Moments: In this method, the theoretical moments of the population and the sample moments are equated correspondingly to deduce the estimators of the parameters. Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form given in equation (2 and 3), we have

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)} \quad B\geq\mu$$

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)} \quad B<\mu$$

This distribution is having three parameters μ, α and β. Hence we equate the first three moments of the population and the sample, which leads to the following equations.

$$\bar{x}=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right) \quad B\geq\mu$$

$$\bar{x}=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right) \quad B<\mu \tag{28}$$

and

$$s^2=\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)^2\right) \quad for\ B\geq\mu$$

$$s^2=\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)^2\right) \quad for\ B<\mu \tag{29}$$

$$\beta_2=\frac{3S_1^2\left(2S_2-S_1^2\right)+S_4-4S_1S_3}{\left(S_2-S_1^2\right)^2} \quad for\ B<\mu$$

$$\beta_2=\frac{3P_1^2\left(2P_2-P_1^2\right)+P_4-4P_1P_3}{\left(P_2-P_1^2\right)^2} \quad for\ B\geq\mu \tag{30}$$

Where $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $s^2=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2$ and $\beta_2=\frac{n\sum_{i=1}^{n}(x_i-\bar{x})^4}{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^2}$

For given values of A, solving the above equations (28), (29) and (30) simultaneously by using Newtons-Raphson method, we can obtain the estimators for the parameters μ, α and β. Sample mean $\overline{X}$ is an unbiased estimator for the parameter μ. Variance of $\overline{X}$ is

$$\mathrm{var}\left(\overline{X}\right)=\mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$

$$=\frac{1}{n}\alpha^2\left[\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)^2\right] \quad for\ B\geq\mu$$

$$=\frac{1}{n}\alpha^2\left[\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)^2\right] \quad for \quad B<\mu \qquad (31)$$

Maximum Likelihood Method of Estimation

Case (i): For $B \geq \mu$

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (3), then the likelihood function of the sample is

$$L=\left(\frac{\beta}{\alpha}\right)^n \prod_{i=1}^{n} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\int_0^\infty e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i + \int_0^{\left(\frac{B-\mu}{\alpha}\right)^\beta} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i} \qquad (32)$$

Taking logarithms on both sides of (32), we get

$$LogL = n\log\beta - n\log\alpha$$

$$-\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta - \log\sum_{i=1}^{n}\left(\int_0^\infty e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i + \int_0^{\left(\frac{B-\mu}{\alpha}\right)^\beta} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i\right) \qquad (33)$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta > 0$, we obtain the estimate of β using the moment method of estimation using the equation (30).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (33) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes (1911)) i.e., when β is odd, we find μ which maximizes log L. From equation (33) Log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta$ is minimum only when μ is the median.

Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero.

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^\beta}{\left(\frac{x_i-\mu}{\alpha}\right)} + \frac{\beta}{\alpha}\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)} = 0 \qquad (34)$$

To derive maximum likelihood estimator of α, consider the derivative of Log L w. r. to α and equate it to zero. This implies

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta + \frac{\beta}{\alpha}\left(\frac{B-\mu}{\alpha}\right)\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)} - \frac{n}{\alpha} = 0 \qquad (35)$$

Solving the equations (30), (34) and (35) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

Case (ii): For $B < \mu$

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (3), then the likely hood function of the sample is

$$L=\left(\frac{\beta}{\alpha}\right)^n \prod_{i=1}^{n} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\int_0^\infty e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i - \int_0^{\left|\frac{B-\mu}{\alpha}\right|^\beta} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i} \qquad (36)$$

Taking logarithms on both sides of (36), we get

$$LogL = n\log\beta - n\log\alpha -$$

$$\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta - \log\sum_{i=1}^{n}\left(\int_0^\infty e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i - \int_0^{\left|\frac{B-\mu}{\alpha}\right|^\beta} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i\right) \qquad (37)$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta > 0$, we obtain the estimate of β using the moment method of estimation using the equation (30).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (37) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution i.e., when β is odd, we find μ which maximizes Log L. From equation. (37) log L is maximum if

$\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero.

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}+\frac{\beta}{\alpha}\frac{\left|\frac{B-\mu}{\alpha}\right|}{\left(\frac{B-\mu}{\alpha}\right)}\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}=0 \quad (38)$$

To derive maximum likelihood estimator of α, consider the derivative of Log L w. r. to α and equate it to zero.

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}+\frac{\beta}{\alpha}\left|\frac{B-\mu}{\alpha}\right|\frac{e^{-\left|\frac{B-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}-\frac{n}{\alpha}=0 \quad (39)$$

Solving the equations (30), (38) and (39) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

Conclusion

In this paper, we have introduced right truncated generalized Gaussian distribution. Generalized Gaussian distribution is useful in analyzing several data sets arising at places at image processing, speech recognition, signal processing, statistical quality control, agricultural experimentation, industrial experimentation and biological experiments. The various distributional properties such as distribution function, moments, skewness, kurtosis, hazard function and survival function are derived. It is observed that the hazard function is sometimes increases and decreases depending upon the truncation parameter. The order statistics of the variate under study are also derived. This distribution is useful for analyzing several data sets in management science, finance, quality control and agricultural experiments. Some inferential aspects of the distribution, method of moments, and maximum likelihood estimation are also derived.

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