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Short Communication

Fixed Point Theorm for Mappings in Bimetric Space

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Abstract

In the present paper, we shall proved a fixed point theorem for mapping in Bimetric space.

Keywords: Bimetric space, continuous mapping, fixed point.

Introduction

In 1968 Maia[1] generalized the results of well known Banach S.[2]contraction Principle by taking two metrics on a set X. During the past few years Maia's theorem was generalized and fixed point theorems proved in several directions by Isekl K.[3], Iyer S.[4], Mishra S.N.[5] Ray B.K.[6], Rus I.A.[7], Singh S.P. and Pant S.P.[8], Bhore S.[9], Phatak H.K.[10] and others.

Our Main Results

Theorem (1):- Let (X,d_1,d_2) be Bimetric space suchthat the following conditions holds: i. $d_1(x,y) \le d_2(x,y)$ for all x,y ε X. ii. (X,d_1) is complete space. iii. $T : X \to X$ be a mapping is continuous with respect to $d_1$and satisfies inequality. iv.

$$d_2(Tx,Ty) \le \alpha \frac{d_2(x,Tx)d_2(y,Ty)}{d_2(x,Ty)+d_2(y,Tx)+d_2(x,y)} + \beta\frac{d_2(x,y)[1+\sqrt{d_2(x,y)d_2(x,Tx)}+\sqrt{d_2(x,y)d_2(y,Tx)}]}{[1+d_2(x,y)+d_2(x,Tx)\,d_2(x,Ty)d_2(y,Tx)d_2(y,Ty)]} + \gamma\frac{[\sqrt{d_2(x,Tx)d_2(y,Ty)}]^2+[\sqrt{d_2(x,Ty)d_2(y,Tx)}]^2}{d_2(x,y)}$$

for all x,y in X where $\alpha, \beta, \gamma \ge 0$ and $\alpha + \beta + \gamma < 1$. Then T have a uniqne fixed point.

Proof: Let x_0 ε X be an arbitrary and define a sequence $\{x_n\}$ by $x_1 = Tx_0$, $x_2 = Tx_1$, $x_3 = Tx_2$,......................... in general $x_n = Tx_{n-1}$, $x_{n+1} = Tx_n$, n = o , 1 , 2 , 3, ...
On using inequality (iv), we have

$$d_2(x_{n+1},x_{n+2}) = d_2(Tx_n, Tx_{n+1})$$

$$\le \alpha\frac{d_2(x_n,x_{n+1})d_2(x_{n+1},x_{n+2})}{d_2(x_n,x_{n+2})+d_2(x_{n+1},x_{n+1})+d_2(x_n,x_{n+1})} + \beta\frac{d_2(x_n,x_{n+1})[1+\sqrt{d_2(x_n,x_{n+1})d_2(x_n,x_{n+1})}+\sqrt{d_2(x_n,x_{n+1})d_2(x_{n+1},x_{n+1})}]}{[1+d_2(x_n,x_{n+1})+d_2(x_n,x_{n+1})\,d_2(x_n,x_{n+2})d_2(x_{n+1},x_{n+1})d_2(x_{n+2},x_{n+1})]} + \gamma\frac{[\sqrt{d_2(x_n,x_{n+1})d_2(x_{n+1},x_{n+2})}]^2+[\sqrt{d_2(x_n,x_{n+2})d_2(x_{n+1},x_{n+1})}]^2}{d_2(x_n,x_{n+1})}$$

$$\le \alpha\frac{d_2(x_n,x_{n+1})d_2(x_{n+1},x_{n+2})}{d_2(x_{n+1},x_{n+2})} + \beta\, d_2(x_n,x_{n+1}) + \gamma\, d_2(x_{n+1},x_{n+2})$$

$$(1-\gamma)\, d_2(x_{2n+1},x_{2n+2}) \le (\alpha+\beta)\, d_2(x_n,x_{n+1})$$

or

$$d_2(x_{n+1},x_{n+2}) \le q d_2(x_n,x_{n+1}) \text{where } q = [\tfrac{\alpha+\beta}{1-\gamma}]$$

Similarly, $d_2(x_n,x_{n+1}) \le q d_2(x_n,x_{n-1})$

Hence $d_2(x_{n+1},x_{n+2}) \le q^{2n+1} d_2(x_0,x_1)$for n = 0, 1, 2, 3, ...
now using the condition (i) we have
$d_1(x_{n+1},x_{n+2}) \le q^{2n+1}\, d_2(x_0,x_1)$
for n = 1, 2, 3, ... Since q <1, it follows that $\{x_n\}$ is a Cauchy sequence in (X, d_1) and therefore by the hypothesis(ii) of completeness of (X, d_1), it has a limit z in (X, d_1).

Since T is continuous with respect to (X, d_1), we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z$$

Similarly, by the continuity of T with respect to d_1 *then* Tz = z.
Therefore z is a common fixed point of T.
Now to show uniqneness of z, let z' be the another common fixed point T. Then from (iv) we have

$$d_2(z,z') = d_2(Tz,Tz') \le \alpha\frac{d_2(z,z)d_2(z',z')}{d_2(z,z')+d_2(z',z)+d_2(z,z')} + \beta\frac{d_2(z,z')\left[1+\sqrt{d_2(z,z')d_2(z,z)}+\sqrt{d_2(z,z')d_2(z',z)}\right]}{[1+d_2(z,z')+d_2(z,z)\,d_2(z,z')d_2(z',z)d_2(z',z')]} + \gamma\frac{[\sqrt{d_2(z,z)d_2(z',z')}]^2+[\sqrt{d_2(z,z')d_2(z',z)}]^2}{d_2(z,z')}$$

$$= \beta\frac{d_2(z,z')[1+d_2(z,z')]}{[1+d_2(z,z')]} + \gamma\, d_2(z,z')$$

$$d_2(z,z') = d_2(Tz,Tz') \le (\beta+\gamma)\, d_2(z,z')$$

Since $(\beta + \gamma) <1$ this gives z = z'. This complete the proof of the theorem.

Theorem (2): Let. (X,d_1,d_2) be a Bimetric space such that the following conditions holds: i. $d_1(x,y) \le d_2(x,y)$ for all x, y, ε X. ii. (X, d_1) is complete space. iii. Two mappings S, T: X→ X are continuous with respect to d_1 and satisfies inequality. iv.

$$d_2(Sx,Ty) \le \alpha\frac{d_2(x,Sx)d_2(y,Ty)}{d_2(x,Ty)+d_2(y,Sx)+d_2(x,y)} + \beta\frac{d_2(x,y)[1+\sqrt{d_2(x,y)d_2(x,Sx)}+\sqrt{d_2(x,y)d_2(y,Sx)}]}{[1+d_2(x,y)+d_2(x,Sx)\,d_2(x,Ty)d_2(y,Sx)d_2(y,Ty)]} + \gamma\frac{[\sqrt{d_2(x,Sx)d_2(y,Ty)}]^2+[\sqrt{d_2(x,Ty)d_2(y,Sx)}]^2}{d_2(x,y)}$$

for all x, y in X where α, β, γ ≥ 0 and (α + β+ γ) < 1. There S and T have a unique common fixed point.

Proof: Let x_0 ε X be an arbitrary and define a sequence $\{x_n\}$ by $x_1 = Sx_0$, $x_2 = Tx_1$,....... in general $x_{2n} = Tx_{2n-1}$, $x_{2n+1} = Sx_{2n}$, ...

On using inequality (iv) we have

$d_2 (x_{2n+1},x_{2n+2}) = d_2 (Sx_{2n} ,Tx_{2n+1})$

$$\leq \alpha\frac{d_2(x_{2n},x_{2n+1})d_2(x_{2n+1},x_{2n+2})}{d_2(x_{2n},x_{2n+2})+ d_2(x_{2n+1},x_{2n+1})+ d_2(x_{2n},x_{2n+1})}$$

$$+ \beta\frac{d_2(x_{2n},x_{2n+1})[1+\sqrt{d_2(x_{2n},x_{2n+1})d_2(x_{2n},x_{2n+1})} + \sqrt{d_2(x_{2n},x_{2n+1})d_2(x_{2n+1},x_{2n+1})}]}{[1+d_2(x_{2n},x_{2n+1})+ d_2(x_{2n},x_{2n+1})\, d_2(x_{2n},x_{2n+2})d_2(x_{2n+1},x_{2n+1})d_2(x_{2n+1},x_{2n+2})]}$$

$$+ \gamma\frac{[\sqrt{d_2(x_{2n},x_{2n+1})d_2(x_{2n+1},x_{2n+2})}]^2+[\sqrt{d_2(x_{2n},x_{2n+2})d_2(x_{2n+1},x_{2n+1})}]^2}{d_2(x_{2n}, x_{2n+1})}$$

$$\leq \alpha\frac{d_2(x_{2n},x_{2n+1})d_2(x_{2n+1},x_{2n+2})}{d_2(x_{2n+1},x_{2n+2})}$$

$$+ \beta\frac{d_2(x_{2n},x_{2n+1})\ [1+d_2(x_{2n},x_{2n+1})]}{[1+d_2(x_{2n},x_{2n+1})]} + \gamma\, d_2 (x_{2n+1},x_{2n+2})$$

$(1-\gamma)\, d_2 (x_{2n+1},x_{2n+2}) \leq\alpha\, d_2 (x_{2n},x_{2n+1}) + \beta\, d_2 (x_{2n} ,x_{2n+1})$

$= (\alpha+\beta)\, d_2 (x_{2n} ,x_{2n+1})$

or $d_2 (x_{2n+1},x_{2n+2}) \leq q\, d_2 (x_{2n} ,x_{2n+1})$

where $q = [\frac{\alpha+\beta}{1-\gamma}] < 1$

Similarly, $d_2 (x_{2n} ,x_{2n+1}) \leq q\, d_2 (x_{2n} ,x_{2n+1})$

Hence $d_2 (x_{2n} ,x_{2n+1}) \leq q^{2n+1} d_2 (x_0 ,x_1)$ for n = 1,2,3, Now using the inequality (i) we have

$d_1 (x_{2n+1} ,x_{2n+2}) \leq q^{2n+1} d_2 (x_0,x_1)$ for n = 1, 2, 3, ...

Since q < 1, it follows that $\{x_n\}$ is a Cauchy sequence in (X,d_1) and therefore by the condition (ii) of completeness of (X,d_1) ,it has a limit z in (X,d_1).

Since S is continuous with respect to (X,d_1), we have

$$z = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = Sz$$

Similarily, by the continuity of T with respect to d_1 then Tz= z. Therefore z is a common fixed point of S and T.

Now to show uniqueness of z, let z' be the another common fixed point of S and T.

Then form (ii) we have,

$d_2 (z,z') = d_2 (Sz, Tz')$

$$\leq\alpha\frac{d_2(z,z)d_2 (z',z')}{d_2(z,z') +d_2(z',z)+ d_2 (z,z')}+$$

$$\beta\frac{d_2(z,z')\left[1+\sqrt{d_2(z,z')d_2(z,z)} + \sqrt{d_2(z,z')d_2(z',z)}\right]}{[1+d_2(z,z')+ d_2(z,z)\ d_2(z,z')d_2(z',z)d_2(z',z')]}$$

$$+ \gamma\frac{[\sqrt{d_2(z,z)d_2(z,z')}]^2 \ +[\sqrt{d_2(z,z')d_2 (z',z)}]^2}{d_2(z,z')}$$

$$+\beta\, \frac{d_2(z,z')[1+d_2(z,z')]}{[1+d_2(z,z')]} + \gamma\, d_2 (z,z')$$

$d_2 (z,z') \leq (\beta + \gamma)\, d_2 (z,z')$

Since (β + γ) <1 this gives z = z'. This completes the proof of the theorem.

Conclusion

In this paper we tried and succeded in changing the inequality used in the mapping. Thus we have extended and generalized the results of Bimetric space.

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