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Response of LTI Systems to Arbitrary Inputs: The Convolution Sum

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Abstract

The major theme of the paper is the characterization of discrete-time signals and systems in time domain. Here we characterized LTI systems by their unit sample response $h(n)$ and derived the convolutions summation, which is the formula for determining the response $y(n)$ of the system characterized by $h(n)$ to any given input response $x(n)$. We also deal some simple methods to determine the convolution.

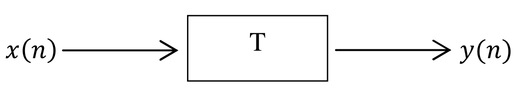
Keywords: Discrete-time signal, LTI systems, unit sample response $h(n)$, input response $x(n)$, output response $y(n)$, convolution.

Introduction

Convolution is a mathematical operation which takes two functions and produces a third function that represents a amount of overlap between one of the function and a reversed and translated version of the other function. How do we predict the response of LTI system to an arbitrary input once we know the impulse response? The secret in convolution[1], Here is the mathematical definition of convolving two functions $x(n)$ and $h(n)$ to create an output$y(n)$[2].

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \text{OR} y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \quad (1)$$

Linear Time-Invariant (LTI) System: A system takes in an input functions and returns an output function.



(Input sequence)(Output sequence)
Figure-1
System

An DT-LTI system is a special type of system. As the name suggests, it must be both linear and time-invariant, as defined below.

Linear: Scaling: $T[ax(n)] = aT[x(n)]$
Superposition: $T[ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$

Time invariant: If$y(n) = T[x(n)]$, then $y(k-n) = T[x(k-n)]$

(Input Impulse) (Impulse response)

Figure-2
Linearn Time invarient System

Characterization of LTI system by means of its impulse response: Let us give $\delta(n)$ unit impulse as the input to a LTI system. The output of this system is called impulse response $h(n)$. Refer figure-2. By the time invariant property of DT-LTI system the response to the applied impulse at any time n_0 is $h(n-n_0) = T[\delta(n-n_0)]$.
(2)

Also by linearity property of superposition, for $x(n_1)\delta(n-n_1) + x(n_2)\delta(n-n_2)$ inputs the corresponding output must be equal to $x(n_1)h(n-n_1) + x(n_2)h(n-n_2)$,

where $x(n_1)$ and $x(n_2)$are amplitudes of $x(n)$ at $n = n_1$ and $n = n_2$ respectively. The discrete input sequence $x(n)$ is represented as $x(n) = - - x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \cdots \ldots\ldots\ldots\ldots. + x(n_0)\delta(n-n_0) + - - -$
OR
$x(n) = T[x(n)] = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$
(may be finite or infinite) (3)

By observing the relation between $\delta(n)$ and $h(n)$ as given in equation (2), the output of DT-LTI system is

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (4)$$

This equation is called convolution sum. It is characterization of DT-LTI system by unit impulse response[4].

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \text{OR } y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \quad (5)$$

Thus DT-LTI system is completely characterized by its impulse response $h(n)$ only and the output $y(n)$ of an DT-LTI system for given input $x(n)$ is obtained by convolving the impulse response $h(n)$ by the input signal $x(n)$[2].

Properties of convolution

Commutative: The convolution of two signals $x(n)$ and $h(n)$ is given by

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k) = h(n) * x(n)$$

Thus $y(n) = x(n) * h(n) = h(n) * x(n)$

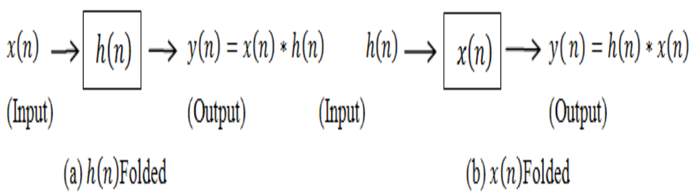


Figure-3
Commutative property

Associative: $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

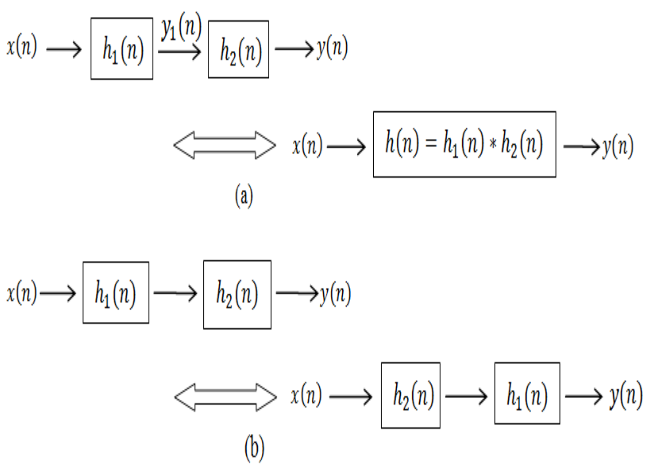


Figure-4
Associative property

LHS: $y_1(n) = x(n) * h_1(n)$
where $x(n)$ is input signal
$y(n) = y_1(n) * h_2(n)$
Where $y_1(n)$ is input signal
$y(n) = [x(n) * h_1(n)] * h_2(n)$ (A)

RHS: $x(n)$ is input signal that applied to an equivalent system $h(n) = h_1(n) * h_2(n)$
$\therefore y(n) = x(n) * h(n) = x(n) * [h_1(n) * h_2(n)]$ (B)
(A) and (B) proves associativity.

Distributive: $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

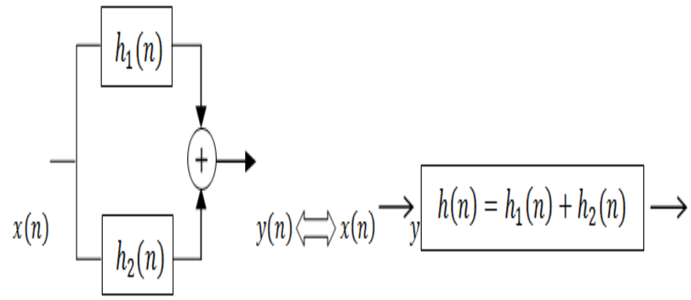


Figure-5
Distribution property

Computation of Linear Convolution

The four basic steps of convolution namely folding, shifting, multiplication and summation are carried out mainly by any of four methods: Tabular, Graphical, Direct and multiplication[2].

Folding: Fold $h(k)$about $k = 0$ to obtain $h(-k)$.
Shifting: Shift $h(-k)$ by n_0 to the right (left) if n_0 is positive (negative) to obtain$h(n_0 - k)$.
Multiplication: Multiply $x(k)$ by $h(n_0 - k)$ to obtain the product$v_{n_0}(k) = x(k)h(n_0 - k)$.
Summation: Sum of the values of product sequences to obtain value of output at $n = n_0 y(k) = \sum v_{n_o}(k)$.

Tabular Method of DT-Linear Convolution: Let us consider
$x(n) = (1, 2, 3, 1)$ be input signal and $h(n) = (1, 2, 1, -1)$ be impulse response. Number of data points L in output $y(n)$is $L = M + N - 1$,
where M = number of data points in $x(n)$, N = number of data points in $h(n)$[2].
Here $M = 4, N = 4\ \therefore L = 4 + 4 - 1 = 7$.
As $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$

Since the time origins of $x(k)$ and $h(k)$ are not same therefore left shift by 1 is carried out at 16th row and $y(-1)$ is computed in 17th row[2].

In this example $\sum x(n) = 7$ and $\sum h(n) = 3$ and $\sum y(n) = 21$
Therefore it satisfies $\sum y(n) = \sum x(n) \cdot \sum h(n)$.

Graphical Representation:

Multiplication Method: This is the simplest method of computing convolution. This method does not reflect directly folding and shifting phenomenon, however multiplication and summation operations are seen clearly. Refer fig. (7). The origin time of $x(n)$ and $h(n)$ are shown by vertical and horizontal arrows. This decides the position of $y(0)$. All diagonals up will show $y(-1)$ etc. and all diagonals down will show $y(1), y(2)$ etc[4].

Table1

$$\boldsymbol{y(n) = \sum x(k)h(n-k)}$$

Time↓Origin															
1	k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	Conv Sym.
2	$x(k)$							1	2	3	1				
3	$h(k)$						1	2	1	-1					
4	$h(-k)$folded w.r.t. origin					-1	1	2	1						
5	$y(0) = \sum_{0}^{1} x(k)h(0-k)$							2	2						4
6	Right shift by 1						-1	1	2	1					
7	$y(1) = \sum_{0}^{2} x(k)h(1-k)$							1	4	3					8
8	Right shift by 1							-1	1	2	1				
9	$y(2) = \sum_{0}^{3} x(k)h(2-k)$							-1	2	6	1				8
10	Right shift by 1								-1	1	2	1			
11	$y(3) = \sum_{1}^{3} x(k)h(3-k)$								-2	3	2				3
12	Right shift by 1									-1	1	2	1		
13	$y(4) = \sum_{2}^{3} x(k)h(4-k)$									-3	1				-2
14	Right shift by 1										-1	1	2	1	
15	$y(5) = \sum_{3}^{3} x(k)h(5-k)$										-1				-1
16	Right shift by 1				-1	1	2	1							
17	$y(-1) = \sum_{0}^{0} x(k)h(-1-k)$							1							1

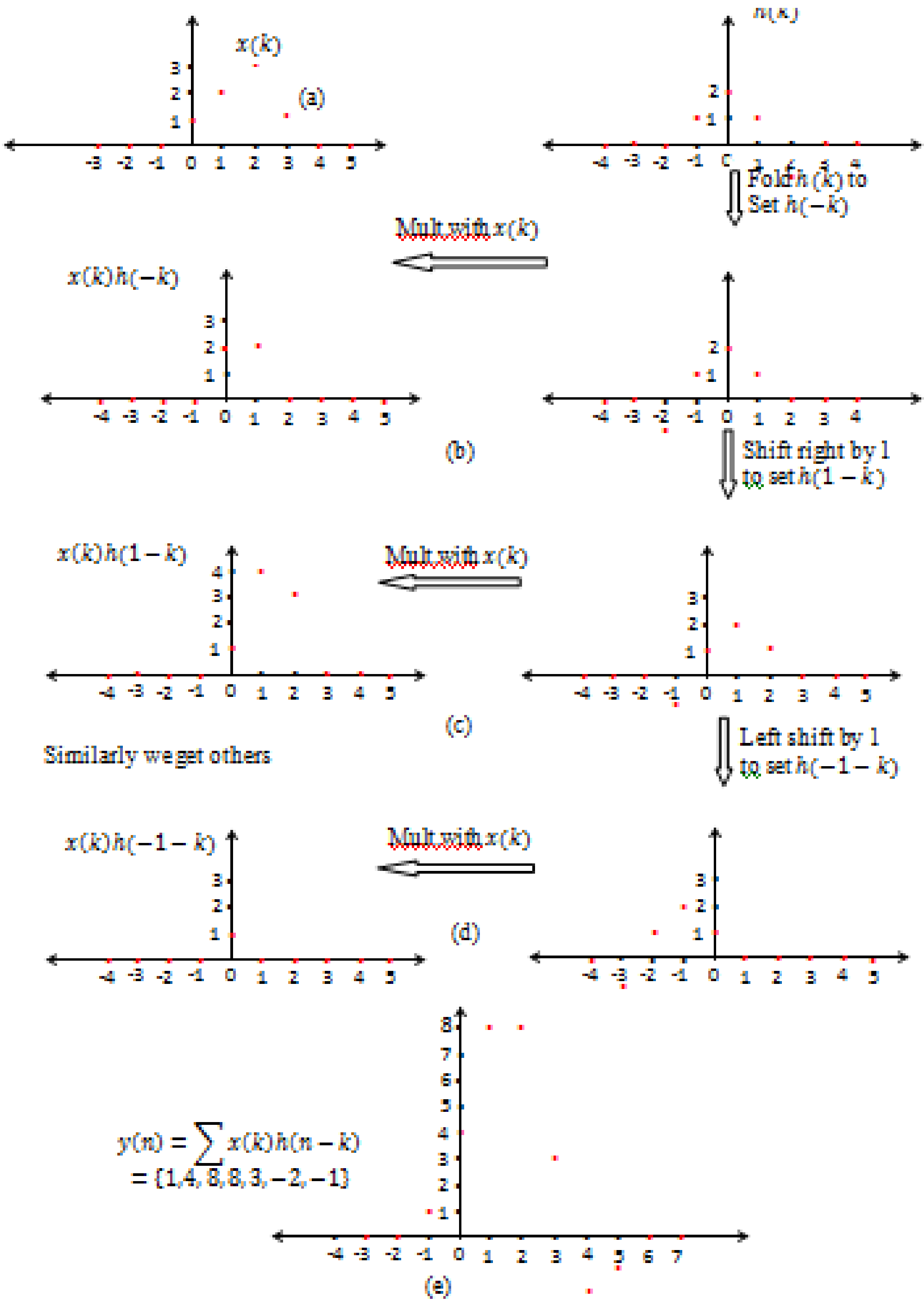


Figure-6

Geometrical representation showing $\sum y(n) = \sum x(n) \sum h(n)$

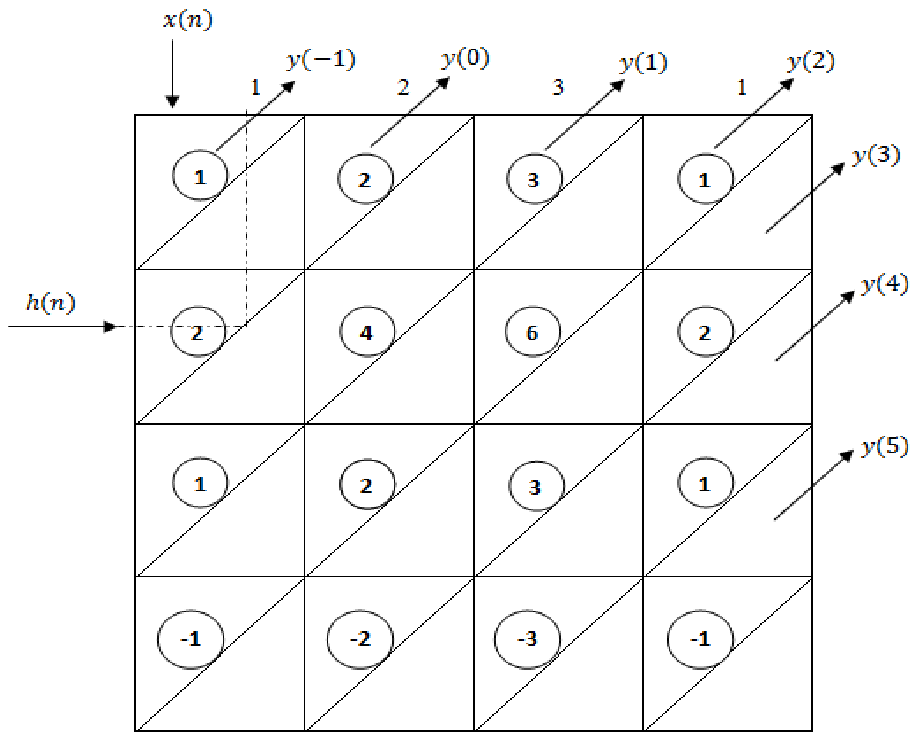


Figure-7
Multi[plication method

At each square, the multiplication of $x(n)$ and $h(n)$ is shown in circle. The diagonal sum of all multiplications will provide the convoluted point $y(n)$.
$y(-1) = 1$
$y(0) = 2 + 2 = 4$
$y(1) = 1 + 4 + 3 = 8$
$y(2) = -1 + 2 + 6 + 1 = 8$

Similarly $y(3) = 3, y(4) = -2, y(5) = -1$
$\therefore y(n) = \{1, 4, 8, 8, 3, -2, -1\}$

Direct Method: In above example at
$n = 0, x(n) = [1, 2, 3, 1], h[-k] = [-1, 1, 2, 1]$
$y(0) = \sum_{k=0}^{1} x(k)h(-k) = x(0)h(0) + x(1)h(-1) = 1 \times 2 + 2 \times 1 = 4.$

Shifting $h(-k)$ right by 1. At $n = 1, h(1-k) = [-1, 1, 2, 1]$
$y(1) = \sum_{k=0}^{2} x(k)h(1-k) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 1 \times 1 + 2 \times 2 + 3 \times 1 = 8.$

Similarly we can find $y(2), y(3), y(4), y(5)$.

Since origin time of $x(n)$ and $h(n)$ are not same, so we will perform left shift by one of the folded $h(k)$. At $n = -1, h(-1-k) = [-1, 1, 2, 1]$

$$y(-1) = \sum_{k=0}^{0} x(k)h(-1-k) = x(0)h(-1) = 1 \times 1 = 1$$
$\therefore y(n) = \{1, 4, 8, 8, 3, -2, -1\}.$

Here also $\sum y(n) = \sum x(n) \cdot \sum h(n)$.

Conclusion

i. Any DT-LTI systems are completely characterized by its unit sample response h(n) and the output is $y(n) = x(n) * h(n)$. ii. The output of any DT-LTI system is a convolution of the inputsignal x(n) with the unit impulse response h(n). i.e. Any DT-LTI $\leftrightarrow y(n) = x(n) * h(n)$.

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