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Combined Representation of Atkinsion and Fredholm Operators in terms of Index

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Abstract

This paper shows that a unique representation between Atkinsion and Fredholm operators with the help of non-negative index, zero and arbitrary index in terms of index except for a rational factor whereas every Atkinsion operator with non-positive index can be represented as a finite dimensional perturbation of a left invertible Atkinsion operator.

Keywords: Atkinsion operator, Fredholm operator, Index.

Introduction

The notion of Atkinson and Fredholm operators which associates either a natural number or + ∞ or -∞ known as index. The operator that in the quotient algebra of all continuous operators with respect to the ideal of all finite dimensional operators they are right or left invertible respectively. Such invertible are known as Atkinson operators[1]. For notion of Atkinson and Fredholm operators, let X be vector space over ϕ, $\Re$ a normal operator of algebra on X, which contain I, F$(\Re)$ be the ideal of all finite dimensional operator in $\Re$, therefore ρ- ideal, $\Re$ the algebra $\Re$/F$(\Re)$ of equivalence classes. Let $\hat{T}$ denotes the equivalence class of an operator T$\in\Re$ with respect to F$(\Re)$. Let A($\Re$)be the set of all Atkinson operator of the algebra $\Re$. Also $\sum(\Re)$ is the set of all Fredholm operator of the algebra $\Re$.

Definitions

Definition-1: Let T$\in\Re$, where $\Re$ is called Atkinson operator (relative to $\Re$), if $\hat{T}$ is left or right invertible.
Definition-2: T$\in\Re$ is called **Fredholm operator[2]** or σ-transformation (relative to $\Re$), if $\hat{T}$ is invertible.
Definition-3: If T$\in L(\mathrm{X})$ and if at least one of the defects $\alpha(\mathrm{T}),\beta(\mathrm{T})$ is finite, then ind (T) = $\alpha(\mathrm{T})-\beta(\mathrm{T})$ is the representative of the index of T. ind (T) is $+\infty$ iff $\alpha(\mathrm{T})=\infty$ and $\beta(\mathrm{T})<\infty$, ind (T) = $-\infty$ iff $\alpha(\mathrm{T})<\infty$ and $\beta(\mathrm{T})=\infty$.

We then say that T possesses an index; T is therefore an Atkinson operator relative to $L(\mathrm{X})$. ind (T) is finite iff $\alpha(\mathrm{T})$ and $\beta(\mathrm{T})$ is finite. So, T possesses a finite index then T is a Fredholm operator relative to $L(\mathrm{X})$.

Theorems

Theorem (1.1): T$\in\Re$ is an Atkinson operator iff is relatively $\Re-$regular and at least one of the defects $\alpha(\mathrm{T}),\beta(\mathrm{T})$ is finite. T$\in\Re$ is Fredholm operator, iff T is relatively $\Re$-regular[1,2] and both the defects are finite.

Theorem (1.2): (Index Theorem): Let S and T$\in L(\mathrm{X})$ are operators with finite indexes. If S and T have also finite indexes, then it can be written as
ind (ST) = ind (S) + ind (T)

Theorem (1.3): (**Generalized index theorem):** If S, T$\in L(\mathrm{X})$ is operators with finite null defects and finite image defects respectively, then it can be expressed as
ind(ST) = ind (S) + ind(T).

Proof: If $\alpha(S)$ *and* $\alpha(T)$ are finite, then S, T$\in \mathbf{A}_{\alpha}(L(\mathrm{X})$ But then S, T lie in $\sum$, and hence the result follows from above definitions and theorems. In the other case these are three possibilities:

$S\in \mathbf{A}_{\alpha}\text{-}\mathbf{A}_{\beta}$, T$\in\sum$
S$\in\sum$, T$\in \mathbf{A}_{\alpha}\text{-}\mathbf{A}_{\beta}$,
S$\in \mathbf{A}_{\alpha}\text{-}\mathbf{A}_{\beta}$, T $\in \mathbf{A}_{\alpha}\text{-}\mathbf{A}_{\beta}$,

From, the definition of defect it follows that ST always belongs to $A_{\alpha}\text{-}A_{\beta}$[3]. Hence it holds every time that
ind (ST) = $-\infty$ and ind (S) + ind (T) = $-\infty$.

One can likewise prove the proposition if S and T belongs to A_β,

Theorem (1.4): If T∈ *L*(X) has an index K and if K∈ $\sum(X)$, then it follows ind (T+K) = ind (T).

Theorem (1.5): If T∈ $\Re$ is an Atkinson operator with $\alpha(T) \le \beta(T)$, i.e. ind (T) ≤ 0 if T= $R+K$, Where $R \in \Re$ possesses a left inverse S ∈ $\Re$ and K∈ $\mathbf{F}(\Re)$

Theorem (1.6): If T∈ $\Re$ is also a Fredholm operator with $\alpha(T) = \beta(T) < \infty$. Then simultaneously fro above (Theorem 1.5) it follows that ind (T) = 0, iff T = R+K, where R∈ $\Re$ possesses an inverse in $\Re$ and K∈ $\mathbf{F}(\Re)$.

Proof: It follows similar to the theorems (1.4) and (1.5), since $\alpha(T) = \beta(T) < \infty$, the operator S can be constructed in the manner as in the proofs of the theorems (1.4) and (1.5).

We now consider the semi-group $\sum(\Re)$.

Let **U** = {ind (T): T∈ $\sum$ }. If **U** is a subgroup of the additive group of integers and corresponding to every n∈ **U** we can choose a definite Fredholm operator with ind (T_n) = n. By using ind (T_n) = n and theorems (1.4) and (1.5). We can easily get results.

Theorem (1.7): Every Fredholm operator T∈ $\Re$ with index (T) = n can be expressed in the form T = T_nR+K,
where R possesses an inverse in $\Re$ and K∈ $\mathbf{F}(\Re)$ conversely every operator T∈ $\Re$ of the above form is a Fredholm operator with index n.

Proof: If T∈ $\sum(R)$ and ind (T) = n, then by using definition (1, 2) and theorem (1.2), it follows that the equivalence class $\hat{T} \in \hat{\Re} = \Re/\mathbf{F}(\Re)$ of T lies in the neben calss of the normalizer $\eta = \{\hat{S} : i(\hat{S}) = 0\}$ of the group **G** of invertible element of $\hat{\Re}$[4]. As the $\hat{T}_k\,\eta$, k∈ **U**, are all neben classes of η, and since $i\ (\hat{T})$ = ind (T) = n, we have $\hat{T} \in \hat{T}_n\eta$, therefore $\hat{T} = \hat{T}\hat{R}_1$ with ind (R_1) = 0. for $R_1 \in \hat{R}_1$. Therefore T = $T_nR_1+K_1$, $K_1 \in \mathbf{F}(\Re)$, and by theorem (1.6) $R_1 = R+K_2$, where R possesses an inverse in $\Re$ and $K_2 \in \mathbf{F}(\Re)$. Then it follows that
T = T_n (R+K_2) + K_1
= T_nR + K, K∈ $\mathbf{F}(\Re)$
The converse follows from theorems (1.4), (1.5) and (1.6).

Some properties of the index: If S and T∈ $\sum(\Re)$, and K∈ $\mathbf{F}(\Re)$ then the properties of the index can be highlighted as follows:
ind (ST) = ind (S) + ind (T)
ind (T+K) = ind (T)
ind (T) = 0, provided T possesses an inverse in $\Re$.

Theorem (1.8): Let d be an integral valued function on $\sum$ with the following properties:
d(ST) = d(S) + d(T)
d(T+K) = d(T), if and only if K∈ $\mathbf{F}(\Re)$
d(T) = 0, if and only if T possesses an inverse of $\Re$.
Then there exists a rational number r such that d (T) = r.ind (T), for all T belonging to $\sum$.

Proof: If $\sum_n = \{T \in \sum$: ind(T) = n} then for T∈ $\sum_n$ we have by theorem (1.7), T= T_nR + K and by the properties (a)-(c) we also have d(T) = d(T_n), hence d is a constant on $\sum_n$ we wish to denote by d respectively. $\overline{ind}$ maps of the set { $\sum_n$ }, defined as follows:
$\bar{d}(E_n) = d(T_n), T_n \in \sum_n$; $\overline{ind}\left(\sum_n\right)$=ind (T_n) = n, $T_n \in \sum_n$ the $\overline{index}$ is one-one [4,5] On account of the index theorem we have $T_nT_m \in \sum_{m+n}$, from which it follows that h = $\bar{d}o\ \overline{ind}^{-1}$ is a homomorphism of the subgroup **U** = {ind (T) :T∈ $\sum$ } of the additive group Z of integers into Z[6,7]. In fact if n, m∈ U, thus h (n + m) = $\bar{d}$ $(\overline{ind}^{-1}(m+n)) = \bar{d}(\sum_{m+n}) = d(T_n.T_m)$[8, 9,10]
= d(T_n) + d(T_m)
$= \bar{d}(\sum_n) + \bar{d}(\sum_m)$
$= \bar{d}(\overline{ind}^{-1}(n)) + \bar{d}(\overline{ind}^{-1}(m))$
= h (n) + h(m)

In **U** there exists a smallest positive number n_0 such that
U = {Kn_0: K∈ Z}

For n∈ **U**, we have n = K(n)n_0.

If T lies in $\sum$, then $T \in \sum_n$ for an $n = K(n)n_0 \in \mathbf{U}$ and it follows that

$$d(T) = \bar{d}(\textstyle\sum_n) = (h \circ \overline{ind})(\textstyle\sum_n)$$
$$= h(\overline{ind}(\textstyle\sum_n))$$
$$= h(n)$$
$$= h(K(n)(n_0))$$
$$= K(n)\, h(n_0)$$
$$= \frac{h(n_0)}{n_0} n$$
$$= \frac{h(n)}{n_0} \text{ ind } (T) = r \text{ ind } (T), \quad r = \frac{h(n)}{n_0}$$

If U = Z, therefore $n_0 = 1$, then d is an integral multiple of index.

Conclusion

Atkinsion operators with non-negative index and Fredholm operators with zero and arbitrary index give a mathematical representation for both operators in terms of index except for a rational factor.

References

1. Atkinsion F.V., On relatively regular operators, *Acta. Sci. Math*., **15**, 38-56 **(1953)**

2. Ruston A.E., Operators with Fredholm theory, *J. Lond. Math. Soc*., **29**, 318-326 **(1954)**

3. Ramm A.G., A simple proof of the Fredholm Alternative and a characterization of the Fredholm operators, *American Mathematical Monthly Journal*, **5,6,7, 81**, **(2001)**

4. Kothe G., Top. Vector Space, I and II, Springer Verlag, 205-225 **(1969)**

5. Goldberg S., Unbounded Linear operators, Mc Graw Hill, New Yorks, 124-140 **(1966)**

6. Robertson A.P and Robertson W., Topological Vector Spaces Cambridge, University Press, 115-137 **(1969)**

7. Taylor A.E., Introduction to functional analysis, New York, Wiley, 79-92 **(1958)**

8. Grotherieck A., Topological Vector Space, *Gordon and Breech*, 62-71 **(1975)**

9. Adasch N., Topological Vector spaces, Lecture notes in Maths, *Springer-Verlag*, 95-108 **(1978)**

10. Meise R. and Vogt D., A characterization of the quasi-normable Frechet spaces, *Math. Nachr.,* **112**, 141-150 **(1985)**