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Iterative Laplace Transform Method for Solving Fractional Heat and Wave-Like Equations

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Abstract

In this paper, we derive the closed form solutions of the fractional heat and wave like equations in terms of Mittag-Leffler functions by the use of iterative Laplace transform method. In the process the time-fractional derivatives are considered in Caputo sense for the said problem.

Keywords: Laplace transform, Iterative Laplace transform method, heat and wave-like equations, Caputo fractional derivative, Mittag-Leffler function, fractional differential equation. **MSC (2010):** 26A33, 33E12, 35R11, 44A10.

Introduction

Fractional calculus has been attracting the attention of scientists and engineers from long time ago, resulting in the development of many applications[1-3]. Various methods for the solution of fractional differential equations are available in literature, including fractional subequation method[4], fractional wavelet method[5-8], fractional Laplace Adomian decomposition method[9,10], fractional operational matrix method[11,12], fractional variational iteration method[13,14], fractional improved homotopy perturbation method[15,16], fractional differential transform method[17] and fractional complex transform method[18].

The iterative method was introduced in 2006 by Daftardar-Gejji and Jafari to solve numerically the nonlinear functional equations[19,20]. By now, the iterative method has been used to solve many integer and fractional boundary value problem[21, 22]. Jafari et al. firstly solved the fractional partial differential equations by the use of iterative Laplace transform method (ILTM)[23]. More recently, Fractional Fokker-Planck equations are solved by the ILTM[24].

For the present problem, we considered the fractional heat and wave-like equations with variable coefficients in the following form:

$$D_t^{\alpha} u = f(x,y,z)\frac{\partial^2 u}{\partial x^2} + g(x,y,z)\frac{\partial^2 u}{\partial y^2} + h(x,y,z)\frac{\partial^2 u}{\partial z^2}, \qquad (1)$$

and initial conditions:

$$u(x,y,z,0) = \hbar(x,y,z),\ u_t(x,y,z,0) = \ell(x,y,z), \qquad (2)$$

where $\alpha (0<\alpha\le 2)$ denotes the fractional derivative. In the case, when $0<\alpha\le 1$, and $1<\alpha\le 2$; then equation. (1) leads to a fractional heat-like and wave-like equations with variable coefficients, respectively.

Preliminaries and Notations

In this section, we give some basic definitions and properties of fractional calculus and Laplace transform theory, which shall be used in this paper:

Definition: The Caputo fractional derivative of function $u(x,t)$ is defined as[25]

$$D_t^{\alpha} u(x,t) = \frac{1}{\left|(q-\alpha)\right.}\int_0^t (t-\upsilon)^{q-\alpha-1} u^{(q)}(x,\upsilon)d\upsilon,\ q-1<\alpha\le q,\ q\in N, \qquad (3)$$

$$= J_t^{q-\alpha} D^q u(x,t).$$

here $D^q \equiv \frac{d^q}{dt^q}$ and J_t^{α} stands for the Riemann-Liouville fractional integral operator of order $\alpha>0$ defined as[26]

$$J_t^{\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\upsilon)^{\alpha-1} u(x,\upsilon)d\upsilon,\ \upsilon>0, (q-1<\alpha\le q), q\in N \qquad (4)$$

Definition: The Laplace transform of a function $\phi(t),\ t>0$ is defined as

$$\mathrm{L}[\phi(t)] = \Phi(s) = \int_0^{\infty} e^{-st}\phi(t)dt. \qquad (5)$$

Definition: Laplace transform of $D_t^{\alpha} u(x,t)$ is given as[27]

$$\mathrm{L}[D_t^{\alpha} u(x,t)] = \mathrm{L}[u(x,t)] - \sum_{k=o}^{q-1} u^k(x,0)s^{\alpha-k-1},\ \ q-1<\alpha\le q,\ q\in N \qquad (6)$$

where $u^k(x,0)$ is the k-order derivative of $u(x,t)$ at $t=0$.

Definition: The Mittag-Leffler function which is a generalization of exponential function is defined as [28]

$$E_{\alpha}(z) = \sum_{q=0}^{\infty} \frac{z^q}{\Gamma(\alpha q+1)} (\alpha\in C,\ \mathrm{Re}(\alpha)>0). \qquad (7)$$

a further generalization of (7) is given in the form [29]

$$E_{\alpha,\beta}(z)=\sum_{q=0}^{\infty}\frac{z^q}{\Gamma(\alpha q+\beta)};(\alpha,\beta\in C,\ R(\alpha)>0,R(\beta)>0). \quad (8)$$

Basic Idea of Iterative Laplace Transform Method

To illustrate the basic idea of this method, we consider a general fractional nonlinear nonhomogeneous partial differential equation with the initial conditions of the form:

$$D_t^{\alpha}u(x,t)+Ru(x,t)+Nu(x,t)=g(x,t), \qquad q-1<\alpha\le q,\ q\in N \quad (9)$$

$$u^k(x,0)=h_k(x),\ k=0,1,2..........,q-1 \quad (10)$$

where $D_t^{\alpha}u(x,t)$ is the Caputo fractional derivative of the function u(x, t), R is the linear differential operator, N represents the general nonlinear differential operator and g(x, t) is the source term. Applying the Laplace transform (denoted by L throughout the present paper) in Equation (9), we get

$$\mathrm{L}[D_t^{\alpha}u(x,t)]+\mathrm{L}[R\,u(x,t)+Nu(x,t)]=\mathrm{L}[g(x,t)]. \quad (11)$$

Using Equation (6), we have

$$\mathrm{L}[u(x,t)]=\frac{1}{s^{\alpha}}\sum_{k=0}^{q-1}s^{\alpha-1-k}u^k(x,0)+\frac{1}{s^{\alpha}}\mathrm{L}[g(x,t)]-\frac{1}{s^{\alpha}}\mathrm{L}[Ru(x,t)+N\,u(x,t)]. \quad (12)$$

Taking inverse Laplace transform of Equation (12) implies

$$u(x,t)=\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{q-1}s^{\alpha-1-k}u^k(x,0)+\mathrm{L}[g(x,t)]\right)\right]-\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathrm{L}[Ru(x,t)+Nu(x,t)]\right], \quad (13)$$

Now we apply the Iterative method,

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t) \quad (14)$$

Since R is a linear operator,

$$R\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=\sum_{i=0}^{\infty}R(u_i(x,t) \quad (15)$$

and the nonlinear operator N is decomposed as

$$N\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=N(u_0(x,t))+\sum_{i=1}^{\infty}\left\{N(\sum_{k=0}^{i}u_k(x,t))-N(\sum_{k=0}^{i-1}u_k(x,t))\right\} \quad (16)$$

Substituting (14), (15) and (16) in (13), we get

$$\sum_{i=0}^{\infty}u_i(x,t)=\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{q-1}s^{\alpha-1-k}u^k(x,0)+\mathrm{L}[g(x,t)]\right)\right]- \mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathrm{L}\left[\sum_{i=0}^{\infty}R(u_i(x,t))+N(u_0(x,t))+\sum_{i=1}^{\infty}\left\{N(\sum_{k=0}^{i}u_k(x,t))-N(\sum_{k=0}^{i-1}u_k(x,t))\right\}\right]\right], \quad (17)$$

We define the recurrence relations as

$$u_0(x,t)=\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{q-1}s^{\alpha-1-k}u^k(x,0)+\mathrm{L}[g(x,t)]\right)\right]$$

$$u_{q+1}(x,t)=-\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathrm{L}\left[R\big(u_q(x,t)\big)-\left\{N(\sum_{k=0}^{q}u_k(x,t))-N(\sum_{k=0}^{q-1}u_k(x,t))\right\}\right]\right],\quad q\ge 1$$

$$u_1(x,t)=-\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathrm{L}\left[R\big(u_0(x,t)\big)+N\big(u_0(x,t)\big)\right]\right] \quad (18)$$

Therefore the q-term approximate solution of (9) - (10) in series form is given by

$$u(x,t)\cong u_0(x,t)+u_1(x,t)+u_2(x,t)+........+u_q(x,t), \qquad q=1,2,....... \quad (19)$$

Applications

In this section, the fractional heat and wave-like equations with variable coefficients are solved by ILTM.

Example: Consider the following one-dimensional fractional heat-like equation:

$$D_t^{\alpha}u(x,t)=\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}, \qquad 0<\alpha\le 1, \quad (20)$$

subject to the initial condition

$$u(x,0)=x^2 \quad (21)$$

Applying the Laplace transform in Equation (20) and making use of (21) we get

$$\mathrm{L}[u(x,t)]=\frac{x^2}{s}+\frac{1}{2s^{\alpha}}x^2\mathrm{L}\left[\frac{\partial^2 u}{\partial x^2}\right] \quad (22)$$

Taking inverse Laplace transform of Equation (22) implies

$$u(x,t)=x^2+\mathrm{L}^{-1}\left[\frac{1}{2s^{\alpha}}x^2\mathrm{L}\left[\frac{\partial^2 u}{\partial x^2}\right]\right] \quad (23)$$

Now, applying the Iterative method, Substituting (14) - (16) into (23) and applying (18), we obtain the components of the solution as follows:

$$u_0(x,t)=u(x,0)=x^2 \quad (24)$$

$$u_1(x,t)=\mathrm{L}^{-1}\left[\frac{1}{2s^{\alpha}}x^2\mathrm{L}\left[\frac{\partial^2 u_0}{\partial x^2}\right]\right]$$

$$=x^2\frac{t^{\alpha}}{\Gamma(\alpha+1)} \quad (25)$$

$$u_2(x,t)=\mathrm{L}^{-1}\left[\frac{1}{2s^{\alpha}}x^2\mathrm{L}\left[\frac{\partial^2(u_0+u_1)}{\partial x^2}\right]\right]-\mathrm{L}^{-1}\left[\frac{1}{2s^{\alpha}}x^2\mathrm{L}\left[\frac{\partial^2 u_0}{\partial x^2}\right]\right]$$

$$=x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)-x^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$=x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad (26)$$

Therefore, the series form solution is given by

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+.............$$

$$u(x,t)=x^2\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+.......................\right]=x^2E_\alpha(t^\alpha) \tag{27}$$

where $E_\alpha(t^\alpha)$ is the Mittag-Leffler function, defined by Equation(7)

Remark: Setting $\alpha=1$, Equation (20) reduced to one-dimensional heat-like equation:

$$\frac{\partial u}{\partial t}=\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2},$$

with solution

$$u(x,t)=x^2e^t. \tag{28}$$

Example: Consider the following Two-dimensional fractional heat-like equation:

$$D_t^\alpha u(x,y,t)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\qquad 0<\alpha\le 1, \tag{29}$$

subject to the initial conditions

$$u(x,y,0)=\sin x\sin y, \tag{30}$$

Applying the Laplace transform in Equation (29) and making use of (30) we get

$$\mathrm{L}\left[u(x,y,t)\right]=\frac{\sin x\sin y}{s}+\frac{1}{s^\alpha}\mathrm{L}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \tag{31}$$

Taking inverse Laplace transform of Equation (31) implies

$$u(x,t)=\sin x\sin y+\mathrm{L}^{-1}\left[\frac{1}{s^\alpha}\mathrm{L}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]\right] \tag{32}$$

Now, applying the Iterative method,
Substituting (14) - (16) into (32) and applying (18), we obtain the components of the solution as follows:

$$u_0(x,y,t)=u(x,y,0)=\sin x\sin y \tag{33}$$

$$u_1(x,y,t)=\mathrm{L}^{-1}\left[\frac{1}{s^\alpha}\mathrm{L}\left[\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right]\right]$$

$$=-2\sin x\sin y\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{34}$$

$$u_2(x,y,t)=\mathrm{L}^{-1}\left[\frac{1}{s^\alpha}\mathrm{L}\left[\frac{\partial^2(u_0+u_1)}{\partial x^2}+\frac{\partial^2(u_0+u_1)}{\partial y^2}\right]\right]-\mathrm{L}^{-1}\left[\frac{1}{s^\alpha}\mathrm{L}\left[\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right]\right]$$

$$=\sin x\sin y\left(\frac{(-2)^2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2t^\alpha}{\Gamma(\alpha+1)}\right)+2\sin x\sin y\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$=(-2)^2\sin x\sin y\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{35}$$

Therefore, the series form solution is given by

$$u(x,y,t)=u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+.............$$

$$u(x,y,t)=\sin x\sin y\left[1+\frac{(-2t^\alpha)}{\Gamma(\alpha+1)}+\frac{(-2t^\alpha)^2}{\Gamma(2\alpha+1)}+.......................\right]=\sin x\sin y\left[E_\alpha(-2t^\alpha)\right] \tag{36}$$

Remark 2. Setting $\alpha=1$, Equation (29) reduced to Two-dimensional heat-like equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},$$

with solution

$$u(x,y,t)=e^{-2t}\sin x\sin y \tag{37}$$

Example 3. Consider the following Three-dimensional fractional heat-like equation:

$$D_t^\alpha u(x,y,z,t)=x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right),\quad 0<\alpha\le 1, \tag{38}$$

Subject to the initial condition

$$u(x,y,z,0)=0, \tag{39}$$

Applying the Laplace transform in Equation (38) and making use of (39) we get

$$\mathrm{L}[u(x,y,z,t)]=\frac{1}{s^\alpha}\mathrm{L}\left(x^4y^4z^4\right)+\frac{x^2}{36}\mathrm{L}\left(\frac{1}{s^\alpha}u_{xx}\right)+\frac{y^2}{36}\mathrm{L}\left(\frac{1}{s^\alpha}u_{yy}\right)+\frac{z^2}{36}\mathrm{L}\left(\frac{1}{s^\alpha}u_{zz}\right) \tag{40}$$

Taking inverse Laplace transform of Equation (40) implies

$$u(x,y,z,t)=\mathrm{L}^{-1}\left[\frac{1}{s^\alpha}\mathrm{L}\left(x^4y^4z^4\right)+\frac{x^2}{36}\mathrm{L}\left(\frac{1}{s^\alpha}u_{xx}\right)+\frac{y^2}{36}\mathrm{L}\left(\frac{1}{s^\alpha}u_{yy}\right)+\frac{z^2}{36}\mathrm{L}\left(\frac{1}{s^\alpha}u_{zz}\right)\right] \tag{41}$$

Now we apply the Iterative method,
Substituting (14) - (16) into (41) and applying (18), we obtain the components of the solution as follows:

$$u_0(x,y,z,t)=x^4y^4z^4\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{42}$$

$$u_1(x,y,z,t)=\mathrm{L}^{-1}\left[\frac{1}{36s^\alpha}x^2\mathrm{L}\left(\frac{\partial^2 u_0}{\partial x^2}\right)+\frac{1}{36s^\alpha}y^2\mathrm{L}\left(\frac{\partial^2 u_0}{\partial y^2}\right)+\frac{1}{36s^\alpha}z^2\mathrm{L}\left(\frac{\partial^2 u_0}{\partial z^2}\right)\right]$$

$$=x^4y^4z^4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{43}$$

$$u_2(x,y,z,t)=\mathrm{L}^{-1}\left[\frac{1}{36s^\alpha}x^2\mathrm{L}\left(\frac{\partial^2(u_0+u_1)}{\partial x^2}\right)+\frac{1}{36s^\alpha}y^2\mathrm{L}\left(\frac{\partial^2 u_0+u_1}{\partial y^2}\right)+\frac{1}{36s^\alpha}z^2\mathrm{L}\left(\frac{\partial^2(u_0+u_1)}{\partial z^2}\right)\right]$$

$$-\mathrm{L}^{-1}\left[\frac{1}{36s^\alpha}x^2\mathrm{L}\left(\frac{\partial^2 u_0}{\partial x^2}\right)+\frac{1}{36s^\alpha}y^2\mathrm{L}\left(\frac{\partial^2 u_0}{\partial y^2}\right)+\frac{1}{36s^\alpha}z^2\mathrm{L}\left(\frac{\partial^2 u_0}{\partial z^2}\right)\right]$$

$$=\left(x^4y^4z^4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+x^4y^4z^4\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right)-\left(x^4y^4z^4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)$$

$$=x^4y^4z^4\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right) \tag{44}$$

Therefore, the series form solution is given by
$u(x,y,z,t)=u_0(x,y,z,t)+u_1(x,y,z,t)+u_2(x,y,z,t)+.............$

$$u(x,y,z,t)=x^4y^4z^4\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+......................\right]=x^4y^4z^4\left[E_{\alpha}\left(t^{\alpha}\right)-1\right]. \qquad (45)$$

Remark: Setting $\alpha=1$, Equation (38) reduced to Three-dimensional heat-like equation:

$$\frac{\partial u}{\partial t}=x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right),$$

with solution

$$u(x,y,z,t)=x^4y^4z^4(e^t-1) \qquad (46)$$

Example: Consider the following one-dimensional fractional wave-like equation:

$$D_t^{\alpha}u(x,t)=\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}, \qquad 1<\alpha\le 2, \qquad (47)$$

subject to the initial conditions

$$u(x,0)=x\,,\; u_t(x,0)=x^2 \qquad (48)$$

Applying the Laplace transform in Equation (47) and making use of (48) we get

$$\mathrm{L}\left[u(x,t)\right]=\frac{x}{s}+\frac{x^2}{s^2}+\frac{x^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u}{\partial x^2}\right) \qquad (49)$$

Taking inverse Laplace transform of Equation (49) implies

$$u(x,t)=x+x^2t+\mathrm{L}^{-1}\left[\frac{1}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u}{\partial x^2}\right)\right] \qquad (50)$$

Now, applying the Iterative method,
Substituting (14) - (16) into (50) and applying (18), we obtain the components of the solution as follows:

$$u_0(x,t)=x+x^2t \qquad (51)$$

$$u_1(x,t)=\mathrm{L}^{-1}\left[\frac{1}{2s^{\alpha}}x^2\mathrm{L}\left(\frac{\partial^2 u_0}{\partial x^2}\right)\right] = x^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \qquad (52)$$

$$u_2(x,t)=\mathrm{L}^{-1}\left[\frac{1}{2s^{\alpha}}x^2\mathrm{L}\left[\frac{\partial^2(u_0+u_1)}{\partial x^2}\right]\right]-\mathrm{L}^{-1}\left[\frac{1}{2s^{\alpha}}x^2\mathrm{L}\left[\frac{\partial^2 u_0}{\partial x^2}\right]\right]$$

$$=x^2\left(\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)-\left(x^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)=x^2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \qquad (53)$$

Therefore, the series form solution is given by

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+.............$$

$$u(x,t)=x+x^2\left[t+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}......................\right]=x+x^2tE_{\alpha,2}(t^{\alpha}) \qquad (54)$$

where $E_{\alpha,\beta}\left(t^{\alpha}\right)$ is generalization form of Mittag-Leffler function, defined by Equation(8).

Remark 4. Setting $\alpha=2$, Equation (47) reduced to one-dimensional wave-like equation:

$$\frac{\partial u}{\partial t}=\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2},$$

with solution

$$u(x,t)=x+x^2\sinh t \qquad (55)$$

Example 5. Consider the following Two-dimensional fractional wave-like equation:

$$D_t^{\alpha}u(x,y,t)=\frac{1}{12}\left(x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}\right), \qquad 1<\alpha\le 2, \qquad (56)$$

Subject to the initial conditions

$$u(x,y,0)=x^4\,,\; u_t(x,y,0)=y^4 \qquad (57)$$

Applying the Laplace transform in Equation (56) and making use of (57) we get

$$\mathrm{L}\left[u(x,y,t)\right]=\frac{x^4}{s}+\frac{y^4}{s^2}+\frac{x^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u}{\partial x^2}\right)+\frac{y^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u}{\partial y^2}\right) \qquad (58)$$

Taking inverse Laplace transform of Equation (58) implies

$$u(x,y,t)=x^4+y^4t+\mathrm{L}^{-1}\left[\frac{x^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u}{\partial x^2}\right)+\frac{y^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u}{\partial y^2}\right)\right] \qquad (59)$$

Now, applying the Iterative method,
Substituting (14) - (16) into (59) and applying (18), we obtain the components of the solution as follows:

$$u_0(x,y,t)=x^4+y^4t \qquad (60)$$

$$u_1(x,y,t)=\mathrm{L}^{-1}\left[\frac{x^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial x^2}\right)+\frac{y^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial y^2}\right)\right]$$

$$=x^4\frac{t^{\alpha}}{\Gamma(\alpha+1)}+y^4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \qquad (61)$$

$$u_2(x,y,t)=\mathrm{L}^{-1}\left[\frac{x^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2(u_0+u_1)}{\partial x^2}\right)+\frac{y^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2(u_0+u_1)}{\partial y^2}\right)\right]$$

$$-\mathrm{L}^{-1}\left[\frac{x^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial x^2}\right)+\frac{y^2}{12s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial y^2}\right)\right] \qquad (62)$$

$$=x^4\left(\frac{t^{\alpha}}{\lfloor(\alpha+1)}+\frac{t^{2\alpha}}{\lfloor(2\alpha+1)}\right)+y^4\left(\frac{t^{\alpha+1}}{\lfloor(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)-x^4\left(\frac{t^{\alpha}}{\lfloor(\alpha+1)}\right)+y^4\left(\frac{t^{\alpha+1}}{\lfloor(\alpha+2)}\right),$$

$$=x^4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+y^4\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}, \qquad (63)$$

Therefore, the series form solution is given by
$u(x,y,t)=u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+.............$

$$u(x,y,t)=x^4\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots\ldots\right]+y^4\left[t+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\ldots\ldots\right]$$

$$=x^4E_{\alpha}\left(t^{\alpha}\right)+y^4tE_{\alpha,2}(t^{\alpha}). \quad (64)$$

Remark 5 Setting $\alpha=2$, Equation (56) reduced to Two-dimensional wave-like equation:

$$\frac{\partial u}{\partial t}=\frac{1}{12}\left(x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}\right),$$

with solution

$$u(x,y,t)=x^4\cosh t+y^4\sinh t. \quad (65)$$

Example 6. Consider the following Three-dimensional fractional wave-like equation:

$$D_t^{\alpha}u(x,y,z,t)=x^2+y^2+z^2+\frac{1}{2}\left(x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right),\quad 1<\alpha\le 2, \quad (66)$$

subject to the initial conditions

$$u(x,y,z,0)=0,\ u_t(x,y,z,0)=x^2+y^2-z^2. \quad (67)$$

Applying the Laplace transform in Equation (66) and making use of (67) we get

$$\mathrm{L}[u(x,y,z,t)]=\left(\frac{x^2+y^2-z^2}{s^2}\right)+\left(\frac{x^2+y^2+z^2}{s^{\alpha}}\right)+\frac{x^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial u}{\partial x^2}\right)+\frac{y^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial u}{\partial y^2}\right)+\frac{z^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial u}{\partial z^2}\right) \quad (68)$$

Taking inverse Laplace transform of Equation (68) implies

$$u(x,y,z,t)=t\left(x^2+y^2-z^2\right)+\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathrm{L}\left(x^2+y^2+z^2\right)+\frac{x^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial u}{\partial x^2}\right)+\frac{y^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial u}{\partial y^2}\right)+\frac{z^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial u}{\partial z^2}\right)\right] \quad (69)$$

Now, applying the Iterative method,
Substituting (14) - (16) into (69) and applying (18), we obtain the components of the solution as follows:

$$u_0(x,y,z,t)=t\left(x^2+y^2-z^2\right)+\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathrm{L}\left(x^2+y^2+z^2\right)\right]$$

$$=t\left(x^2+y^2-z^2\right)+\left(x^2+y^2+z^2\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \quad (70)$$

$$u_1(x,y,z,t)=\mathrm{L}^{-1}\left[\frac{x^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial x^2}\right)+\frac{y^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial y^2}\right)+\frac{z^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial z^2}\right)\right]$$

$$=\left(x^2+y^2-z^2\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\left(x^2+y^2+z^2\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad (71)$$

$$u_2(x,y,z,t)=\mathrm{L}^{-1}\left[\frac{x^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2(u_0+u_1)}{\partial x^2}\right)+\frac{y^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2(u_0+u_1)}{\partial y^2}\right)+\frac{z^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2(u_0+u_1)}{\partial z^2}\right)\right]$$
$$-\mathrm{L}^{-1}\left[\frac{x^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial x^2}\right)+\frac{y^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial y^2}\right)+\frac{z^2}{2s^{\alpha}}\mathrm{L}\left(\frac{\partial^2 u_0}{\partial z^2}\right)\right]$$

$$=\left(x^2+y^2-z^2\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\left(x^2+y^2+z^2\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \quad (72)$$

Therefore, the series form solution is given by

$$u(x,y,z,t)=u_0(x,y,z,t)+u_1(x,y,z,t)+u_2(x,y,z,t)+\ldots\ldots\ldots\ldots$$

$$u(x,y,z,t)=\left(x^2+y^2-z^2\right)\left[t+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\ldots\ldots\right]+\left(x^2+y^2+z^2\right)\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\ldots\right]$$

$$=t\left(x^2+y^2-z^2\right)E_{\alpha,2}\left(t^{\alpha}\right)+\left(x^2+y^2+z^2\right)\left[E_{\alpha}(t^{\alpha})-1\right]. \quad (73)$$

Remark 6. Setting $\alpha=2$, Equation (66) reduced to Three-dimensional wave-like equation:

$$\frac{\partial u}{\partial t}=x^2+y^2+z^2+\frac{1}{2}\left(x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right),$$

with solution

$$u(x,y,z,t)=\left(x^2+y^2\right)e^t+z^2e^{-t}+\left(x^2+y^2-z^2\right). \quad (74)$$

Conclusion

The solutions of the fractional heat and wave like equations in terms of Mittag-Leffler functions by the use of iterative Laplace transform method were derived. The solutions are obtained in series form that rapidly converges in a closed exact formula with simply computable terms. The calculations are simple and straightforward. The method was tested on six examples on different situations.

References

1. Baleanu D., Diethelm K., Scalas E. and Trujillo J.J., Fractional Calculus, vol. 3 of Series on Complexity, Nonlinearity and Chaos, *World Scientific, Singapore,* **(2012)**
2. Ortigueira M.D., Fractional calculus for scientists and engineers, *Springer ,* **(2011)**
3. Sabatier J., Agrawal O.P. and Tenreiro Machado J.A., Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, *Springer,***(2007)**
4. Zhang S. and Zhang H.Q., Fractional sub-equation method and its applications to nonlinear fractional PDEs, *Physics Letters A,* **375(7)**, 1069–1073 **(2011)**
5. Lepik U., Solving fractional integral equations by the Haar wavelet method, *Applied Mathematics and Computation,***214(2)**, 468–478 **(2009)**
6. Li Y., Solving a nonlinear fractional differential equation using Chebyshev wavelets, *Communications in Nonlinear Science and Numerical Simulation,* **15(9)**, 2284–2292, **(2010)**
7. Rehman M. and Ali Khan R., The Legendre wavelet method for solving fractional differential equations, *Communications in Nonlinear Science and Numerical Simulation,***16 (11)**, 4163–4173 **(2011)**

8. Wu J.L., A wavelet operational method for solving fractional partial differential equations numerically, *Applied Mathematics and Computation,* **214(1)**, 31–40 **(2009)**

9. Jafari H., Khalique C.M., and Nazari M., Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion-wave equations, *Applied Mathematics Letters,* **24(11)**, 1799–1805 **(2011)**

10. Ongun M.Y., The Laplace adomian decomposition method for solving a model for HIV infection of CD4+ cells, *Mathematical and Computer Modelling,* **53(5-6)**, 597–603 **(2011)**

11. Liu Y. and Sun N, Numerical solution of fractional differential equations using the generalized block pulse operational matrix, *Computers and Mathematics with Applications,* **62(3)**, 1046–1054 **(2011)**

12. Saadatmandi A. and Dehghan M., A new operational matrix for solving fractional-order differential equations, *Computers and Mathematics with Applications,* **59(3)**, 1326–1336 **(2010)**

13. Das S., Analytical solution of a fractional diffusion equation by variational iteration method, *Computers and Mathematics with Applications,***57(3)**, 483–487 **(2009)**

14. Sweilam N.H., Khader M.M. and Al-Bar R.F., Numerical studies for a multi-order fractional differential equation, *Physics Letters A,* **371(1-2)**, 26–33 **(2007)**

15. Liu Y., Approximate solutions of fractional nonlinear equations using homotopy perturbation transformation method, *Abstract and Applied Analysis,***2012, Article ID 752869**, 14 **(2012)**

16. Liu Y., Variational homotopy perturbation method for solving fractional initial boundary value problems, *Abstract and Applied Analysis,***2012, Article ID 727031,** 10 **(2012)**

17. Erturk V.S. and Momani S., Solving systems of fractional differential equations using differential transform method, *Journal of Computational and Applied Mathematics,* **215(1)**, 142–151 **(2008)**

18. Ibrahim R.W., Fractional complex transforms for fractional differential equations, *Advances in Difference Equations,* 192 **(2012)**

19. Daftardar-Gejji V. and Jafari H., An iterative method for solving nonlinear functional equations, *Journal of Mathematical Analysis and Applications,* **316(2)**, 753–763 **(2006)**

20. Jafari H., Iterative Methods for solving system of fractional differential equations [Ph.D. thesis], Pune University, **(2006)**

21. Bhalekar S. and Daftardar-Gejji V., Solving evolution equations using a new iterative method, *Numerical Methods for Partial Differential Equations,* **26(4)**, 906–916 **(2010)**

22. Daftardar-Gejji V. and Bhalekar S., Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, *Computers andMathematics with Applications,* **59(5)**, 1801–1809 **(2010)**

23. Jafari H., Nazari M., Baleanu D. and Khalique C.M., A new approach for solving a system of fractional partial differential equations, *Computers and Mathematics with Applications,* **66(5)**, 838–843**(2013)**

24. Yan L., Numerical Solutions of Fractional Fokker-Planck Equations Using Iterative Laplace Transform Method, *Abstract and Applied Analysis,***2013, Article ID 465160**, 7 pages, **(2013)**

25. Caputo M, Elasticita e Dissipazione, *Zani-Chelli, Bologna,*(**1969)**

26. Podlubny I., Fractional Differential Equations, vol. 198, *Academic Press, New York, NY, USA,* **(1999)**

27. Kilbas A.A., Srivastava H.M. and Trujillo J.J., Theory and Applications of Fractional Differential Equations, *Elsevier, Amsterdam,*(**2006)**

28. Miller K.S. and Ross B., An Introduction to the Fractional Calculus and Fractional Differential Equations, *John Wiley and Sons, New York ,USA,***(1993)**

29. Wiman A., Uber de fundamental satz in der theorie der funktionen $E_\alpha(x)$, *Acta Math.,* **29,** 191-201, **(1905)**