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Estimation of Confidence Intervals of a GHROC Curve in the Presence of Scale and Shape Parameters

S. Balaswamy and R. Vishnu Vardhan
Department of Statistics, Pondicherry University, Puducherry 605014, INDIA

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Abstract

Receiver operating characteristic (ROC) Curves are used to assess the accuracy of a diagnostic test in terms of Area under the ROC Curve (AUC). The present work focuses on constructing confidence intervals for an ROC Curve which is derived from two generalized distributions, named as GHROC Curve. Another objective of the paper is to highlight the importance of shape parameter in the explaining the true accuracy of a test. Simulation studies are conducted to demonstrate the proposed methodology using different combinations of scale and shape parameters with various sample sizes.

Keywords: ROC Curve, AUC, confidence intervals and generalized distributions.

Introduction

Receiver Operating Characteristic (ROC) Curve analysis was developed during second world war as a statistical tool in signal detection theory to analyze radar devices in differentiating signal from noise[1]. In recent years, the ROC methodology has been extensively used in diverse areas of research such as Banking, Finance, Engineering, Machine Learning and Medical Sciences. In particular, Leo Lusted[2] introduced the concept of ROC Curve in medicine for analysis of radiographic images. In medical studies, ROC Curve is an extremely useful tool applied in classification problems associated with diagnostic tests. This methodology also provides an assessment of the accuracy of the resulting classifier.

In statistical decision theory, ROC Curve analysis is a well know classification tool which helps in allocating the subjects into one of the known two populations. The entire process of allocation is done using a threshold and at every threshold, a pair of co-ordinates gets generated, namely, 1-Specificity and Sensitivity. The tradeoff between these two co-ordinates gives rise to a unit square plot, named as Receiver Operating Characteristic (ROC) Curve. The co-ordinates are usually referred as the basic intrinsic measures of the ROC curve. However, there is a need to explain how accurate a test is? To address this a well know metric is used in the ROC context namely, Area under the Curve (AUC), where AUC is a summary measure which provides the accuracy of a diagnostic test[3]. AUC can take values between 0 and 1 with practical lower bound value as 0.5 (chance line). So the entire analysis and interpreting the efficiency of a diagnostic test purely depends on sensitivity, specificity and AUC. As it is defined above that the ROC Curve is a function of 1-specificity and sensitivity, these can be expressed using the following notations:

Let x(t) and y(t) denote false positive rate (FPR or 1-Specificity) and true positive rate (TPR or Sensitivity) which can be expressed as

$$x(t) = 1 - F(t) \text{ and } y(t) = 1 - G(t)$$

where F and G are the distribution functions of the populations without and with condition, respectively. We classify observations with scores above a threshold ‘t’ as positive (with condition or abnormal), otherwise as negative (without condition or normal).

Using the information of x(t) and y(t), we can define the expression of ROC which is as follows[4],

$$\text{ROC}(t) = 1 - G\left[F^{-1}(1 - x(t))\right]$$

In recent past, the ROC models so far developed are based on the normal distribution, where both normal and abnormal populations approximate to normal, hence the name Bi-normal ROC Curve [5, 6, 7]. But in practical situations, the data of both the populations may not follow normal and may follow some other distributions. In medical, engineering and life studies, data tend to have extended tails, in this situation, the conventional Bi-normal ROC Curve fails to explain the hidden accuracy of the test considered. Recently, Balaswamy et al.[8] addressed this issue and developed a Hybrid ROC (HROC) Curve which is based on Half Normal and Exponential distributions. However, this model is restricted by considering only scale parameter to illustrate the accuracy. But there are other statistical measures which accounts for the information about the tail property of the data. In this paper, a generalized version of the HROC Curve is proposed by considering the Generalized Half Normal[9] and Generalized Exponential[10] distributions with both scale and shape parameters corresponding to normal as well as abnormal populations. Along with this, 95 % confidence interval constructed for the measures of the proposed ROC model, supported by bootstrap method. Further, the proposed methodology is demonstrated using simulation studies.

Methodology

Let $x_1, x_2 \in S$, be the test scores, which are observed in normal or healthy (H) and abnormal or diseased (D) populations, respectively. Here, it is assumed that H and D populations follow Generalized Half Normal (GHN) and Generalized Exponential (GE) distributions with shape and scale parameters as $\alpha > 0$ *and* $\sigma > 0$ respectively. The probability density function and cumulative distribution function of GHN and GE are given as follows:

$$f(x_1,\alpha,\sigma)=\sqrt{\frac{2}{\pi}}\left(\frac{a}{x_1}\right)\left(\frac{x_1}{\sigma}\right)^{\alpha}\exp\left(-\frac{1}{2}\left(\frac{x_1}{\sigma}\right)^{2\alpha}\right) \tag{1}$$

$$F(x_1,\alpha,\sigma)=1-2\Phi\left[-\left(\frac{x_1}{\sigma}\right)^{\alpha}\right] \tag{2}$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution.

$$g(x_{2,},\alpha,\sigma)=\frac{\alpha}{\sigma}\left(1-\exp\left(-\frac{x_2}{\sigma}\right)\right)^{\alpha-1}\exp\left(-\frac{x_2}{\sigma}\right) \tag{3}$$

$$G(x_2,\alpha,\sigma)=\left(1-\exp\left(-\frac{x_2}{\sigma}\right)\right)^{\alpha} \tag{4}$$

Using the probabilistic definitions, the expressions for x(t) and y(t) are as follows:

$$x(t)=P(S>t\,|\,H)=2\left[1-\Phi\left(\frac{t}{\sigma_H}\right)^{\alpha_H}\right] \tag{5}$$

The expression for FPR is obtained using (2), the TPR expression using (4) is given as

$$y(t)=P(S>t\,|\,D)=1-\left(1-\exp\left(-\frac{t}{\sigma_D}\right)\right)^{\alpha_D}$$

$$y(t)=1-\left[1-\exp\left(-\beta\left[\Phi^{-1}\left(1-\frac{x(t)}{2}\right)\right]^{\frac{1}{\alpha_H}}\right)\right]^{\alpha_D} \tag{6}$$

Here $t=\sigma_H\left[\Phi^{-1}\left(1-\frac{x(t)}{2}\right)\right]^{1/\alpha_H};\ \beta=\frac{\sigma_H}{\sigma_D};\ \zeta=\frac{\alpha_D}{\alpha_H}$ and $\alpha_H, \sigma_H, \alpha_D$ and σ_D are the shape and scale parameters of the GHN and GE distributions, respectively. The expression in (6) is referred as *Generalized Hybrid ROC (GHROC) Curve*, since it is combination of both GHN and GE distributions. It is also showed that the GHROC Curve always has a positive slope, monotonically increasing with false positive rate and invariant under strictly increasing transformations (for proofs see Appendix I).

If the shape parameter values of both H and D populations attains a unit value (i.e., $\alpha_H = \alpha_D = 1$), then the GHROC Curve in (6) reduces to the Hybrid ROC Curve[8] and is

$$y(t)=\exp\left(-\beta\left[\Phi^{-1}\left(1-\frac{x(t)}{2}\right)\right]\right)$$

In ROC methodology, the statistical measure which helps in explaining the overlapping area and the accuracy of a classifier is the Area under the Curve (AUC). It can be interpreted as the probability that a subject randomly selected from the group with condition will have discrimination score indicating greater suspicion than that of a randomly selected subject from the group without condition[11]. It is defined as,

$$AUC=\int_0^1 y(t)\,dt$$

$$AUC=\int_0^1 1-\left[1-\exp\left(-\beta\left[\Phi^{-1}\left(1-\frac{x(t)}{2}\right)\right]^{\frac{1}{\alpha_H}}\right)\right]^{\alpha_D} dx(t) \tag{7}$$

The above expression has no closed form; hence it has to be evaluated using numerical integration.

Confidence Intervals for AUC: The $100(1-\alpha)\%$ confidence interval for AUC can be defined as

$$\widehat{AUC} \pm Z_{1-\frac{\alpha}{2}}\sqrt{Var\left(\widehat{AUC}\right)} \tag{8}$$

where $Z_{1-\frac{\alpha}{2}}$ is the $1-\frac{\alpha}{2}$ standard normal percentile and $Var\left(\widehat{AUC}\right)$ is the estimated variance of $\widehat{AUC}$ which is obtained using bootstrapping. Let 'B' be the number of bootstraps obtained from the data with the sample sizes n_H and n_D respectively from H and D populations. Then the bootstrapped estimate of AUC and its variance are

$$\widehat{AUC}=\frac{1}{B}\sum_{b=1}^{B}AUC_b \tag{9}$$

$$Var\left(\widehat{AUC}\right)=\frac{1}{B-1}\sum_{b=1}^{B}\left(AUC_b-\widehat{AUC}\right)^2 \tag{10}$$

Confidence Intervals for GHROC Curve: The $100(1-\alpha)\%$ confidence intervals for the GHROC Curve are estimated using delta method. This confidence interval for the ROC Curve represents the range at each point of false positive rate and its corresponding true positive rate. Therefore, the $100(1-\alpha)\%$ confidence intervals for false positive rate (FPR) and true positive rate (TPR) are as follows,

$$\widehat{FPR} \pm Z_{1-\frac{\alpha}{2}}\sqrt{Var\left(\widehat{FPR}\right)}$$

$$\widehat{TPR} \pm Z_{1-\frac{\alpha}{2}}\sqrt{Var\left(\widehat{TPR}\right)}$$

where the variance of false positive rate and true positive rate are as follows,

$$Var\left(\widehat{FPR}\right)=\left(\frac{\partial FPR}{\partial \sigma_H}\right)^2 Var\left(\widehat{\sigma_H}\right)+\left(\frac{\partial FPR}{\partial \alpha_H}\right)^2 Var\left(\widehat{\alpha_H}\right) \quad (11)$$

$$Var\left(\widehat{TPR}\right)=\left(\frac{\partial TPR}{\partial \sigma_D}\right)^2 Var\left(\widehat{\sigma_D}\right)+\left(\frac{\partial TPR}{\partial \alpha_D}\right)^2 Var\left(\widehat{\alpha_D}\right) \quad (12)$$

Further, the partial differentiations of FPR and TPR with respect to their parameters are:

$$\frac{\partial FPR}{\partial \sigma_H}=\left(\frac{2\alpha_H t^{\alpha_H}}{\sigma_H^{\alpha_H+1}}\right)\phi\left(\frac{t}{\sigma_H}\right)^{\alpha_H}$$

$$\frac{\partial FPR}{\partial \alpha_H}=\left(\frac{-2t^{\alpha_H}}{\sigma_H^{\alpha_H}}\right)\phi\left(\frac{t}{\sigma_H}\right)^{\alpha_H}\log\left(\frac{t}{\sigma_H}\right)$$

$$\frac{\partial TPR}{\partial \sigma_D}=\alpha_D\,\sigma_D\,e^{-\frac{t}{\sigma_D}}\left(1-\exp\left(-\frac{t}{\sigma_D}\right)\right)^{\alpha_D-1}$$

$$\frac{\partial TPR}{\partial \alpha_D}=-\left(1-\exp\left(-\frac{t}{\sigma_D}\right)\right)^{\alpha_D}\log\left(1-\exp\left(-\frac{t}{\sigma_D}\right)\right)$$

Now, by substituting the above partial derivatives in equations (11) and (12), the expressions for variance of false positive rate and true positive rate are

$$Var\left(\widehat{FPR}\right)=\left(\left(\frac{2\alpha_H t^{\alpha_H}}{\sigma_H^{\alpha_H+1}}\right)\phi\left(\frac{t}{\sigma_H}\right)^{\alpha_H}\right)^2 Var\left(\widehat{\sigma_H}\right)+$$
$$\left(\left(\frac{-2t^{\alpha_H}}{\sigma_H^{\alpha_H}}\right)\phi\left(\frac{t}{\sigma_H}\right)^{\alpha_H}\log\left(\frac{t}{\sigma_H}\right)\right)^2 Var\left(\widehat{\alpha_H}\right) \quad (13)$$

$$Var\left(\widehat{TPR}\right)=\left(\alpha_D\,\sigma_D\,e^{-\frac{t}{\sigma_D}}\left(1-\exp\left(-\frac{t}{\sigma_D}\right)\right)^{\alpha_D-1}\right)^2 Var\left(\widehat{\sigma_D}\right)+ \quad (14)$$
$$\left(-\left(1-\exp\left(-\frac{t}{\sigma_D}\right)\right)^{\alpha_D}\log\left(1-\exp\left(-\frac{t}{\sigma_D}\right)\right)\right)^2 Var\left(\widehat{\alpha_D}\right)$$

The bootstrapped estimates and its variances of the parameters $\sigma_H, \alpha_H, \sigma_D$ *and* α_D are as follows

$$\widehat{\sigma_H}=\frac{1}{B}\sum_{b=1}^{B}\sigma_{H_b}\,,\quad Var\left(\widehat{\sigma_H}\right)=\frac{1}{B-1}\sum_{b=1}^{B}\left(\sigma_{H_b}-\widehat{\sigma_H}\right)^2$$

$$\widehat{\alpha_H}=\frac{1}{B}\sum_{b=1}^{B}\alpha_{H_b}\,,\quad Var\left(\widehat{\alpha_H}\right)=\frac{1}{B-1}\sum_{b=1}^{B}\left(\alpha_{H_b}-\widehat{\alpha_H}\right)^2$$

$$\widehat{\sigma_D}=\frac{1}{B}\sum_{b=1}^{B}\sigma_{D_b}\,,\quad Var\left(\widehat{\sigma_D}\right)=\frac{1}{B-1}\sum_{b=1}^{B}\left(\sigma_{D_b}-\widehat{\sigma_D}\right)^2$$

$$\widehat{\alpha_D}=\frac{1}{B}\sum_{b=1}^{B}\alpha_{D_b}\,,\quad Var\left(\widehat{\alpha_D}\right)=\frac{1}{B-1}\sum_{b=1}^{B}\left(\alpha_{D_b}-\widehat{\alpha_D}\right)^2$$

The expressions in (13) and (14) will help us to plot the lower and upper confidence intervals of the GHROC Curve.

In the next section, illustration on the proposed methodology is given using simulation studies at various sample sizes.

Results and Discussion

Simulation studies are conducted with different combinations of scale and shape parameters of both normal and abnormal populations and the entire simulations are done at various sample sizes {50, 100, 200, 300 and 500}. At every parameter combination and sample size, the AUC and its confidence intervals are obtained. The main purpose of conducting simulations is to show how the AUC of GHROC possesses different values as the scale and shape parameters of the normal and abnormal distributions change. The variations in the parameter values of both populations are used to explain the overlapping area in terms of AUC, this mean that as higher the AUC, lesser will be the overlapping area and vice versa. Further, to demonstrate the behavior of AUC, the entire simulation work is carried out with three different experiments. In first experiment, the shape parameter of abnormal population is varied by fixing the other parameters as constant; in second experiment, the scale parameter of abnormal population is varied by fixing the other parameters as constant and in the third experiment, the scale parameter values of both populations are given a unit value and the shape parameters of both populations are varied. The results so obtained from these experiments are reported in table-1.

In first experiment, when $\alpha_D = 0.8$ with $\sigma_H=1, \sigma_D=1.5, \alpha_H=0.6,$ the AUC is observed to be around 0.6 (60% of accuracy) and as α_D takes higher values as 1 and 1.5, the AUC is observed to have a better value indicating high level of accuracy. Thus, this reflects the scenario as the discrepancy between shape parameters of both normal and abnormal population's increases; AUC attains a larger value indicating a better extent of correct classification with minimum percentage of over lapping area. Suppose, if we have real data set with these parameter values then that particular test will provide a better accuracy. Along with the shape, scale parameter also influences the measure AUC. Further, in second experiment, arbitrary values are fixed for $\alpha_D, \alpha_H, \sigma_H$ and by varying σ_D. This resulted in giving out another interesting point that along with moderate level of discrepancy in shape values, scale parameters also has its influence in explaining the extent of accuracy. As σ_D attains a larger value, the AUC of GHROC tend to have better values than that of the first experiment. So this reveals that along with discrepancy in shape parameters of both populations, if scale parameter tends to explain better variability in the data then it gives rise to good phenomenon to talk about the exact performance of the test considered. However, there is a need to verify that how the accuracy of a test can be given when there is a discrepancy in shape parameters with a unit variability in both populations (i.e., $\sigma_H = \sigma_D = 1$). This is addressed by conducting the third experiment. Here, in the first part of this experiment the shape parameters are varied by taking unit variability and the second part is defined by considering unit values to all the scale and shape parameters of both populations. The results of the first part outline the observation that even though scale parameters are of unit value, the discrepancy in shape values tends to explain the hidden accuracy and as larger the discrepancy between the shape values of two populations better explanation about the accuracy of the test can be given. The second part reveals the finding that when all parameters are made to unit value, then two populations get overlapped giving rise to have AUC nearer to 0.5. Thus, from three experiments it is noticed that shape parameter has its major influence in explaining better accuracy of a test than observed with scale parameter alone. However, scale parameter also has its role in explaining the accuracy and it should not be neglected. The proposed methodology overcomes the limitation of the work proposed by Balaswamy et. al.[8], where they have attempted to explain the accuracy of the test by taking into the account of scale parameters of two populations only. The entire experimentation is graphically visualized in figure-1, by depicting all sorts of scale and shape parameter combinations.

Table-1
AUC and its Confidence Intervals for various combinations of scale and shape parameters at different sample sizes

Experiment	Parameter Values		Sample Size				
			50	100	200	300	500
I	$\sigma_H = 1$, $\sigma_D = 1.5$, $\alpha_H = 0.6$	$\alpha_D = 0.8$	0.6464 (0.5422, 0.7506)	0.6017 (0.5122, 0.6912)	0.6025 (0.5492, 0.6559)	0.5796 (0.5340, 0.6253)	0.6104 (0.5791, 0.6418)
		$\alpha_D = 1$	0.6406 (0.5341, 0.7471)	0.6383 (0.5604, 0.7161)	0.6366 (0.5884, 0.6848)	0.6649 (0.6284, 0.7013	0.6367 (0.6005, 0.6729)
		$\alpha_D = 1.5$	0.7641 (0.6695, 0.8587)	0.7387 (0.6639, 0.8135)	0.7332 (0.6866, 0.7799)	0.7255 (0.6875, 0.7636)	0.7323 (0.7065, 0.7581)
II	$\sigma_H = 1$, $\alpha_H = 0.7$, $\alpha_D = 2.5$	$\sigma_D = 1.5$	0.8163 (0.7339, 0.8986)	0.8115 (0.7472, 0.8758)	0.8433 (0.8065, 0.8800)	0.8445 (0.8194, 0.8696)	0.8333 (0.8103, 0.8563)
		$\sigma_D = 2.25$	0.8876 (0.8345, 0.9408)	0.8843 (0.8444, 0.9243)	0.9066 (0.8837, 0.9295)	0.9008 (0.8803, 0.9213)	0.9011 (0.8872, 0.9150)
		$\sigma_D = 3.5$	0.936 (0.9120, 0.9599)	0.944 (0.9282, 0.9598)	0.9412 (0.9302, 0.9522)	0.9342 (0.9221, 0.9463)	0.9331 (0.9241, 0.9422)
III	$\sigma_H = \sigma_D = 1$	$\alpha_H = 1.5$, $\alpha_D = 2$	0.7034 (0.6220, 0.7847)	0.6935 (0.6275, 0.7595)	0.6575 (0.6067, 0.7084)	0.6912 (0.6564, 0.7260)	0.6808 (0.6534, 0.7082)
		$\alpha_H = 2.17$, $\alpha_D = 3.5$	0.8243 (0.7418, 0.9069)	0.8402 (0.7927, 0.8876)	0.8269 (0.7866, 0.8673)	0.8189 (0.7881, 0.8498)	0.8247 (0.8014, 0.8480)
		$\alpha_H = \alpha_D =$	0.5109 (0.3983, 0.6235)	0.4957 (0.4213, 0.5700)	0.483 (0.4255, 0.5405)	0.5203 (0.4726, 0.5679)	0.5294 (0.4911, 0.5676)

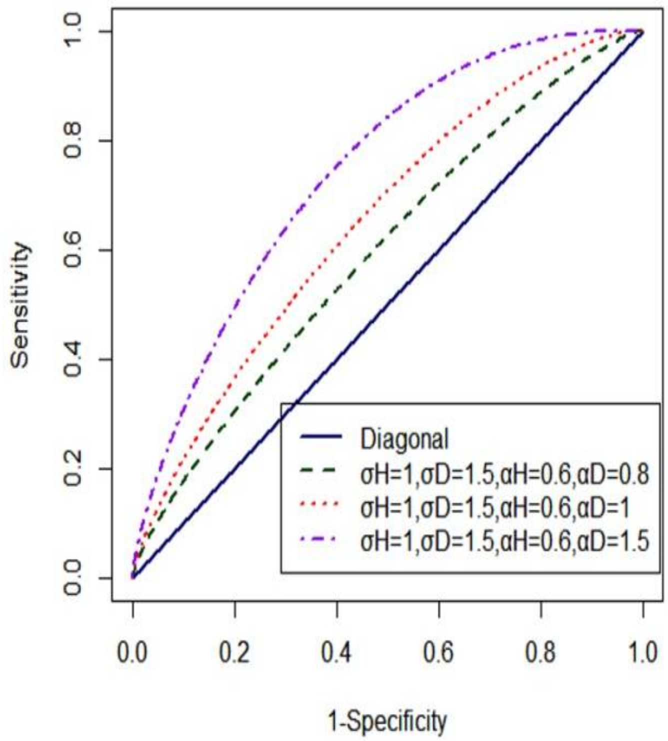

Figure-1a: Effect of Shape Parameter on shaping the GHROC Curve

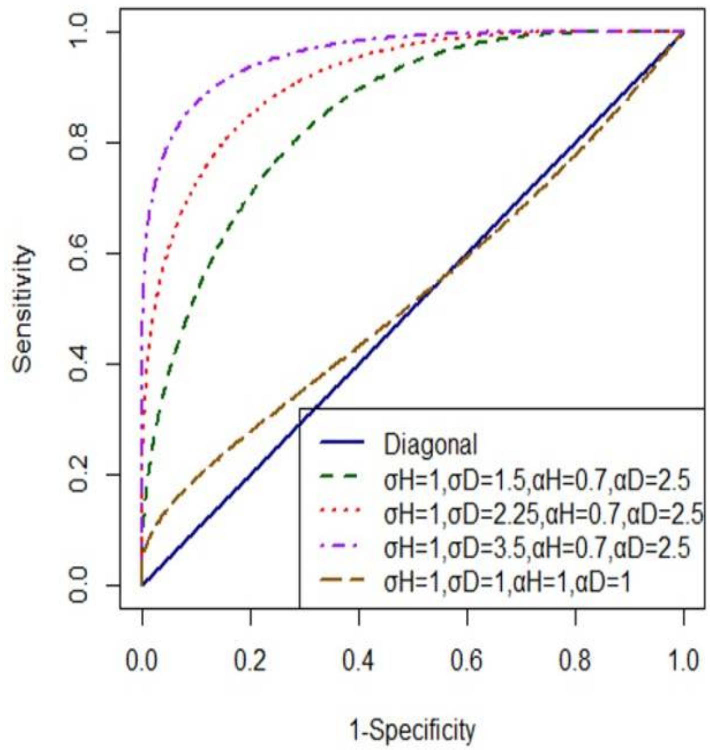

Figure-1b: Effect of Scale Parameter on shaping the GHROC Curve

Figure-1
GHROC Curves at various combinations of scale and shape parameters of both normal and abnormal populations

Apart from explaining the importance and the influence of the scale and shape parameters in ROC context, it is essential to construct the confidence intervals for the measures of ROC Curve. This attempt is to illustrate the changing behavior of the estimates of the proposed ROC Curve. In statistical literature, the theory of interval estimation has gained its importance over point estimation because it reveals the true information of the estimate within the potential uncertainties. Hence, it is very important to address the position of the true estimate in the presence of sample size within the range of potential uncertainties. The $100(1-\alpha)\%$ confidence intervals are constructed for all the combinations which are defined as three different experiments.

Two salient features are explained from the obtained results. First, the perception about the impact of sample size on the width of the confidence intervals and second, the graphical visualization of the true estimates of GHROC Curve along with its confidence intervals. With respect to the first point, it is evident that the sample size effect can be witnessed in terms of the width of the confidence interval. From this, it is noticed that the true estimate is independent from the effect of sample size and its corresponding confidence interval possesses a narrowing down phenomenon. This suggests that as larger the sample size, smaller is the width of the confidence interval. These simulation studies points out the information that irrespective of the sample size and width of the confidence interval, the information about the true estimate of the ROC Curve lie within the potential uncertainties. Even though this is a generally observed phenomenon but the fact to be noticed is that the variability in the populations will get diminished as the sample size takes a larger number, giving rise to a shortened confidence interval.

Further, the confidence intervals are drawn for the measures sensitivity and specificity and are graphically visualized in figure-2. This means that the sensitivity can be obtained at a particular value of specificity and vice versa from these confidence intervals.

Figure-2 clearly explains the confidence intervals for GHROC Curve at various combinations of scale and shape parameters of both populations at a particular sample size. Along with this the lower and upper confidence intervals explain the range of false positive rates and true positive rates. Further, the optimal threshold is also depicted in figure-2 along with the pair (FPR, TPR) obtained at that particular optimal threshold which helps in classifying the subjects/individuals into one of the two populations with better accuracy.

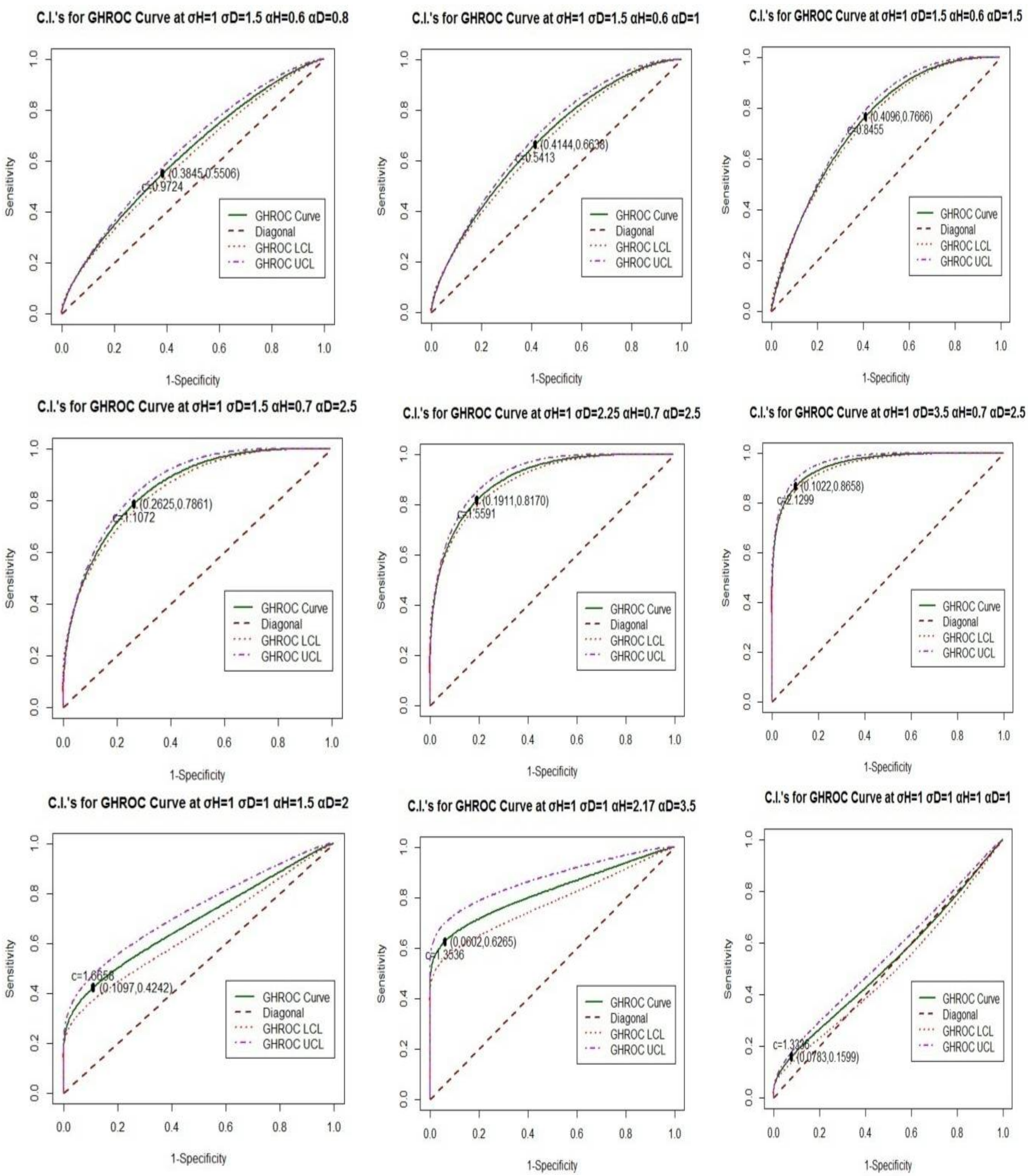

Figure-2
The Confidence Intervals for GHROC Curve at various combinations of scale and shape parameters of both populations at a particular sample size

Conclusions

The present paper is focused on addressing the practical issue where the populations with and without condition under lie two different distributions which are of skewed in nature. Further, confidence intervals expressions are derived for the measures of the proposed ROC Curve. The main objective of the paper is to show the importance of shape parameter in explaining the extent of better accuracy of a test. Simulation studies are conducted to demonstrate the influence of the shape parameter along with the presence of scale parameter. However, in statistical theory, the conclusions cannot be drawn based on the true estimates of the curve, hence; there is a need to provide the potential uncertainties usually referred as confidence intervals for the measures of the curve, which helps in illustrating the tendency of the true estimate within the intervals. The entire exercise is done using three experiments and the effect of sample size is also noted. Further, it is observed that the width of the confidence interval is affected by the size of the sample in turn providing shortened confidence intervals as sample size is considered to be large. Moreover from the proposed methodology it is feasible to identify the sensitivity at a specific false positive rate and vice versa.

Appendix I: Properties of GHROC Curve

Property 1: Generalized Hybrid ROC Curve is monotonically increasing

Proof: Let us consider two false positive values P_1 *and* P_2 such that $P_1 < P_2$ *and* $\Phi^{-1}(\cdot)$ be a strictly increasing function.

Since $P_1 < P_2$ which implies that $1-\frac{p_1}{2} > 1-\frac{p_2}{2}$

$$\Rightarrow \Phi^{-1}\left[1-\frac{p_1}{2}\right] \geq \Phi^{-1}\left[1-\frac{p_2}{2}\right]$$

$$\Rightarrow \exp\left\{-\frac{\sigma_H}{\sigma_D}\left[\Phi^{-1}\left(1-\frac{p_1}{2}\right)\right]^{\frac{1}{\alpha_H}}\right\} \leq \exp\left\{-\frac{\sigma_H}{\sigma_D}\left[\Phi^{-1}\left(1-\frac{p_2}{2}\right)\right]^{\frac{1}{\alpha_H}}\right\}$$

$$\Rightarrow \left[1-\exp\left\{-\frac{\sigma_H}{\sigma_D}\left[\Phi^{-1}\left(1-\frac{p_1}{2}\right)\right]^{\frac{1}{\alpha_H}}\right\}\right]^{\alpha_D} \geq \left[1-\exp\left\{-\frac{\sigma_H}{\sigma_D}\left[\Phi^{-1}\left(1-\frac{p_2}{2}\right)\right]^{\frac{1}{\alpha_H}}\right\}\right]^{\alpha_D}$$

$$\Rightarrow 1-\left[1-\exp\left\{-\frac{\sigma_H}{\sigma_D}\left[\Phi^{-1}\left(1-\frac{p_1}{2}\right)\right]^{\frac{1}{\alpha_H}}\right\}\right]^{\alpha_D} \leq 1-\left[1-\exp\left\{-\frac{\sigma_H}{\sigma_D}\left[\Phi^{-1}\left(1-\frac{p_2}{2}\right)\right]^{\frac{1}{\alpha_H}}\right\}\right]^{\alpha_D}$$

$$\Rightarrow ROC(p_1) \leq ROC(p_2)$$

Hence, the Generalized Hybrid ROC (model) Curve is monotonically increasing.

Property 2: Slope of the Generalized Hybrid ROC Curve equals the likelihood ratio and is positive.

Proof: The derivative of ROC curve at a given pair of coordinates equals the likelihood ratio. Let us parameterize x and y in terms of 't' and the derivative can be written as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(t)}{f(t)}$$

The derivatives of the cumulative distribution functions F(t) and G(t) are the probability distribution functions f(t) and g(t). Therefore the derivative of the Generalized Hybrid ROC Curve is

$$\frac{dy}{dx} = \frac{\frac{\alpha_D}{\sigma_D}\left(1-\exp\left(-\frac{t}{\sigma_D}\right)\right)^{\alpha_D-1}\exp\left(-\frac{t}{\sigma_D}\right)}{\sqrt{\frac{2}{\pi}}\left(\frac{\alpha_H}{\sigma_H^{\alpha_H}}\right)t^{\alpha_H-1}\exp\left(-\frac{1}{2}\left(\frac{t}{\sigma_H}\right)^{2\alpha_H}\right)}$$

On simplifying the above equation, we have

$$\therefore \frac{dy}{dx} = \sqrt{\frac{\pi}{2}}\left(\frac{\alpha_D\sigma_H^{\alpha_H}}{\sigma_D\alpha_H}\right)t^{1-\alpha_H}\left(1-\exp\left(-\frac{t}{\sigma_D}\right)\right)^{\alpha_D-1}\exp\left(\frac{1}{2}\left(\frac{t}{\sigma_H}\right)^{2\alpha_H}-\frac{t}{\sigma_D}\right) \geq 0$$

which is the ratio of the distribution of abnormal scores to normal scores of the two probability densities at the value of 't' and is positive. This is referred as the likelihood ratio of Generalized Hybrid ROC Curve.

Property 3: The Generalized Hybrid ROC Curve is invariant under strictly increasing transformation.

Proof: Let 'S' denote the set of scores with $S \subset \Re$ and $h(\cdot)$ is strictly increasing function. Let $a, b \in S$ *and* $a < b$, then by using strictly increasing function, we can write $h(a) < h(b)$.

The transformed random variables U and V from the respective normal and abnormal classes are

$$P(U \leq t) = P[h(U) \leq h(t)] \;\&\; P(V \leq t) = P[h(V) \leq h(t)]$$

Let us consider the points $(x^*(t), y^*(t))$ on the ROC Curve for the transformed scores

$$x^*(t) = P[h(U) > h(t) | H] = 1 - P[h(U) \leq h(t)] = 1 - P(U \leq t) = x(t)$$

$$y^*(t) = P[h(V) > h(t) | D] = 1 - P[h(V) \leq h(t)] = 1 - P(V \leq t) = y(t)$$

Thus the Generalized Hybrid ROC Curve is invariant to transformation.

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