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Characterization of Uniform Distribution through Expectation of Function of Order Statistics

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Abstract

For characterization of uniform distribution one needs any arbitrary non constant function only in place of approaches such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy etc. available in the literature. Recently Bhatt characterized negative exponential distribution through expectation of non constant function of random variable. Attempt is made to extend the characterization of negative exponential distribution through expectation of any arbitrary non constant function of order statistics.

Keyword: Characterization; Uniform distribution. **MSC 2010 Subject Clasification :** 62E10

Introduction

Characterizations theorem are located on borderline between probability theory and mathematical statistics. It is of general interest to mathematical community, to probabilists and statistician as well as to researchers and practitioner industrial engineering and operation research and various scientist specializing in natural and behavior science, in particular those who are interested in foundation and application of probabilistic model building. (see basic book on characterizations by Lukacs and Laha[1] and the more advance comprehensive mathematical tools (entirely toward normal distribution) see kagan, Linnik and Rao[2]).

Various approaches for characterization of uniform distribution are available in the literature. It is well known that smaller and the larger of a random sample of size two are positively correlate and coefficient of correlation is less or equal to one half. Bartoszyn'ski[3] proposed that a result of this type might exist in connection with a problem in cell division. Since the two daughter cells cannot always be

This work is supported by UGC Major Research Project No: F.No.42-39/2013 (SR), dated 12-3-2013. Distinguished later, the times till their further division can only be recorded as the earlier event and the later event. The correlation between these ordered pairs thus may provide the only information on the independence of the two events. Terreel[4] showed that the coefficient of correlation is one half if and only if random sample comes from rectangular distribution. Terreel's proof is computational nature and use properties of Legendre polynomial. Lopez –Bldzquez[5] gave ease proof for Terreel's characterization and obtained shaper bound on the coefficient of correlation.

Geary[6] stated that given sample of size $n \geq 2$ independent observations come from some distribution on the line then sample mean and variance are independent if and only if observations are normally distributed. The need of some regularity condition for Geary's characterization of normal distribution have been removed by successive refinement (see kagan, Linnik and Rao[2] page. 103). Similar characterization for uniform distribution by Kent[7] asserted that if $n \geq 2$ i.i.d random angels from distribution defined by density on circle, sample mean direction and resultant length are independent if and only if angels come from uniform distribution.

Uniform distribution $U(0,1)$ is neatly characterized by two moment conditions: $E[Max(X_1, X_2)] = 2/3$ and $E(X_1^2) = 1/3$ by Lin[8]. Using two suitable moments of order statistics Too[9] characterize uniform and exponential distribution. Other contributions concerning use of property of maximal correlation coefficient between order statistics, of identically distributed spacings etc [see Stapleton[10], Arnold[11], Driscoll[12], Shimizu[13], Abdelhamid[14]].

Huang[15] studied density estimation by wavelet-based reproducing kernels and further doing error analysis for bias reduction in a spline-ased multi resolution, Chow[16] characterized uniform distribution $U(0,1)$ via moments of n-fold convolution modulo one.

Inequality of Chernoff[17,18] assert that if X is normally distributed with mean 0 and variance 1 and if g is absolutely continuous and g(X) has finite variance, then $E[\{g'(X)\}^2] \geq V[g(X)]$ and equality holds if and only $g(X)$ is linear. Chernoff proved this result using Hermite polynomials where as Chris[19] proved inequality of Chernoff by using Cauchy- Schwarz inequality and Fubini's theorem. Sumrita[20] studied Chernoff-type inequalities for distributions on [-1,1] having symmetric unimodal densities and gave characterization of uniform distributions by inequalities of chernoff-type.

Using identity and equality of expectation of function of order statistics, this research note provides path breaking new characterization of uniform distribution with probability density function (pdf)

$$f(x,\theta) = \begin{cases} \frac{1}{\theta}\ ;\ \mathrm{a} < x < \theta < b; \\ \\ 0\ ;\ otherwise. \end{cases} \qquad \ldots(1.1)$$

where $-\infty \leq a < b \leq \infty$ are known constants and $\left(\frac{1}{\theta}\right)$ is everywhere differentiable function. Since range is truncated by θ from right $a = 0$.

The aim of the present research note is to give a new characterization through expectation of function of order statistics, $\phi(.)$ for uniform distribution. The characterization theorem given in section 2 and section 3 is devoted to applications for illustrative purpose.

Characterization

Theorem: Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_\mathrm{n}$ be a random sample of size n from distribution function F**.** Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ be the corresponding set of order statistics. Assume that F is continuous on the interval (a,b), where $-\infty \leq a < b \leq \infty$. Let $\phi(X_{n:n})$ and $g(X_{n:n})$ be two distinct differentiable and intregrable functions of n^{th} order statistic; $X_{n:n}$, on the interval (a,b), where $-\infty \leq a < b \leq \infty$ and moreover $g(X_{n:n})$ be non-constant of $X_{n:n}$**.** Then

$$\mathrm{E}\left[g(X_{n:n}) + \left(\frac{X_{n:n}}{\mathrm{n}}\right)\frac{d}{dX_{n:n}}g(X_{n:n})\right] = \mathrm{g}(\theta) \qquad \ldots (2)$$

is the necessary and sufficient condition for pdf $f(x,\theta)$ of F to be $f(x,\theta)$ defined in (1).

Proof : Given $f(x,\theta)$ defined in (1), for necessity of (2) if $\phi(X_{n:n})$ is such that $g(\theta) = E[\phi(X_{n:n})]$ where $g(\theta)$ is differentiable function then using $f(x_{n:n},\theta)$; pdf of n^{th} order statistics one gets

$$\mathrm{g}(\theta) = \int_{\mathrm{a}}^{\theta} \phi(\mathrm{x}_{\mathrm{n:n}})\, \mathrm{f}(\mathrm{x}_{\mathrm{n:n}},\theta)\, \mathrm{dx}_{\mathrm{n:n}} \qquad (3)$$

Differentiating with respect to θ on both sides of (2.2), replacing $X_{n:n}$ for θ and simplifying the result one gets

$$\phi(X_{n:n}) = g(X_{n:n}) + \left(\frac{X_{n:n}}{\mathrm{n}}\right)\frac{d}{dX_{n:n}}g(X_{n:n}) \qquad (4)$$

which establishes necessity of (2). Conversely given (2), let $k(x_{n:n},\theta)$ be such that

$$g(\theta) = \int_a^{\theta}\left[g(X_{n:n}) + \left(\frac{X_{n:n}}{\mathrm{n}}\right)\frac{d}{dX_{n:n}}g(X_{n:n})\right] k(x_{n:n},\theta)\, dx_{n:n}, \qquad (5)$$

Since $(1/x_{n:n})^n$ is decreasing integrable and differentiable function on the interval (a,b) with $\mathrm{a}^{\mathrm{n}} = 0$ the following identity holds.

$$g(\theta) \equiv \left(\frac{1}{\theta}\right)^n \int_a^{\theta}\left[\frac{d}{dx_{n:n}}\{x_{n:n}^n g(x_{n:n})\}\right] dx_{n:n} \qquad (6)$$

Differentiating $\{x_{n:n}^n g(x_{n:n})\}$ with respect to $x_{n:n}$ and simplifying (6) after taking $\left\{\frac{d}{dx_{n:n}}x_{n:n}^n\right\}$ as one factor, one gets (6) as

$$g(\theta) \equiv \int_a^{\theta}\left[g(x_{n:n}) + \frac{x_{n:n}^n}{\frac{d}{dx_{n:n}}\{x_{n:n}^n\}}\frac{d}{dx_{n:n}}g(x_{n:n})\right]\left\{\left(\frac{1}{\theta}\right)^n \frac{d}{dx_{n:n}}\{x_{n:n}^n\}\right\} dx_{n:n} \qquad (7)$$

and substituting derivative of $\frac{d}{dx_{n:n}}x_{n:n}^n$ in (2.6) one gets (7) as

$$g(\theta) \equiv \int_{\theta}^{b} \phi(x_{n:n})\left\{n\left(\frac{1}{\theta}\right)^n x_{n:n}^{n-1}\right\} dx_{n:n} \qquad (8)$$

where $\phi(X_{n:n})$ is derived in (4).

By uniqueness theorem from (5) and (8)

$$k(x_{n:n},\theta) = n\left(\frac{1}{\theta}\right)^n x_{n:n}^{n-1}. \qquad (9)$$

Since $(1/x_{n:n})^n$ is decreasing increasing integrable and differentiable function on the interval (a,b) where $-\infty \leq a < b \leq \infty$ and since $\left[\frac{d}{dx_{n:n}}x_{n:n}^n\right]$ is positive intregrable function on the interval (a,b) where $-\infty \leq a < b \leq \infty$ with $a^n = 0$ and integrating (9) over the interval (a,θ) on both sides, one gets (9) as

$$\mathrm{k}(x_{n:n};\theta) = \left(\frac{1}{\theta}\right)^{\mathrm{n}} \frac{\mathrm{d}}{\mathrm{dx}_{\mathrm{n:n}}}\mathrm{x}_{\mathrm{n:n}}^{\mathrm{n}}\ ;\ \mathrm{a} < x_{n:n} < \theta < b \qquad (10)$$

and

$$1 = \int_{\mathrm{a}}^{\theta} \mathrm{k}(x_{n:n};\ \theta)\, \mathrm{d}x_{n:n}. \qquad (11)$$

The equation (10), $[\mathrm{k}(x_{n:n};\theta)]_{\mathrm{n}=1}$ reduces to $\mathrm{f}(\mathrm{x};\theta)$ defined in (1) which establishes sufficiency of (2).

Remark 2.1 Denoting

$$M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}}g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n})} \qquad (12)$$

one can determine $f(x,\theta)$ given in (1) as

$$f(x,\theta) = \left[\frac{\frac{d}{dx_{n:n}}(T(x_{n:n}))}{T(\theta)}\right]_{n=1} \quad (13)$$

where $T(X_{n:n})$ is decreasing function for $-\infty \leq a < b \leq \infty$ with $T(a) = 0$ such that it satisfies

$$M(X_{n:n}) = \frac{d}{dX_{n:n}}\left[\log\left(T(X_{n:n})\right)\right]. \quad (14)$$

Remark. 2.2 The theorem 2.1 for function of n[th] order statistics also holds for function of random variable X when $(n = 1)$.

Examples

Examples: Consider the uniformly minimum variance unbiased estimator, $\widehat{\theta^r}$ of θ^r,

$$g(X_{n:n}) = \left(\frac{n+r}{n}\right) X_{n:n}^r = \widehat{\theta^r}$$

$$\phi(X_{n:n}) = g(X_{n:n}) + \left(\frac{X_{n:n}^n}{\frac{d}{dX_{n:n}}X_{n:n}^n}\right)\frac{d}{dX_{n:n}}g(X_{n:n}) = \left(1+\frac{r}{n}\right)^2 X_{n:n}^r$$

$$M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}}g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n})} = \frac{n}{X_{n:n}}$$

$$\frac{d}{dX_{n:n}}\left[\log(x_{n:n}^n)\right] = \frac{n}{X_{n:n}} = M(X_{n:n})$$

$$T(X_{n:n}) = x_{n:n}^n$$

$$f(x,\theta) = \left\{\frac{\frac{d}{dX_{n:n}}(T(X_{n:n}))}{T(\theta)}\right\}_{n=1} = \frac{1}{\theta}$$

Examples: Using the uniformly minimum variance unbiased (UMVU) estimator $\hat{g}(\theta)$ and maximum likelihood estimator (MLE) $\tilde{g}(\theta)$ of $g(\theta)$ such as mean; θ, r[th] moment; θ^r, e^θ, e^θ, p[th] quantile; $Q_p(\theta)$, distribution function; $F(t)$; reliability function; $\bar{F}(t)$, hazard rate; $\boldsymbol{\lambda}(t)$, one gets $[\phi(X_{n:n}) - g(X_{n:n})]$ as given below

$\mu_1'(\theta)$	$\widehat{\mu_1'(\theta)} = \frac{X_{n:n}}{2}\left[1+\frac{1}{n}\right]$	$\frac{X_{n:n}}{2n}\left[1+\frac{1}{n}\right]$
	$\widetilde{\mu_1'(\theta)} = \frac{X_{n:n}}{2}$	$\frac{X_{n:n}}{2n}\left[1+\frac{1}{n}\right]$
$\mu_r'(\theta)$	$\widehat{\mu_r'(\theta)} = \frac{X_{n:n}^r}{r+1}\left[1+\frac{r}{n}\right]$	$\frac{rX_{n:n}^r}{n(r+1)}\left[1+\frac{r}{n}\right]$
	$\widetilde{\mu_r'(\theta)} = \frac{X_{n:n}^r}{r+1}$	$\frac{r}{n(r+1)}X_{n:n}^r$
e^θ	$\widehat{e^\theta} = \left[1+\frac{X_{n:n}}{n}\right]e^{X_{n:n}}$	$\frac{X_{n:n}}{n}e^{X_{n:n}}\left[1+\frac{X_{n:n}}{n}+\frac{1}{n}\right]$
	$\widetilde{e^\theta} = e^{X_{n:n}}$	$\frac{X_{n:n}}{n}e^{X_{n:n}}$
$e^{-\theta}$	$\widehat{e^{-\theta}} = \left[1-\frac{X_{n:n}}{n}\right]e^{-X_{n:n}}$	$\frac{X_{n:n}}{n}e^{-X_{n:n}}\left[\frac{X_{n:n}}{n}-\frac{1}{n}-1\right]$
	$\widetilde{e^{-\theta}} = e^{-X_{n:n}}$	$-\frac{X_{n:n}}{n}e^{X_{n:n}}$
$Q_p(\theta)$	$\widehat{Q_p}(\theta) = \left[1+\frac{1}{n}\right]X_{n:n}p$	$\left[1+\frac{1}{n}\right]\left(\frac{X_{n:n}}{n}\right)p$
	$\tilde{Q}_p(\theta) = X_{n:n}p$	$\left(\frac{X_{n:n}}{n}\right)p$
$F(t)$	$\hat{F}(t) = \left(\frac{n-1}{n}\right)\frac{t}{X_{n:n}}$	$-\left(\frac{n-1}{n^2}\right)\frac{t}{X_{n:n}}$
	$\tilde{F}(\theta) = \frac{t}{X_{n:n}}$	$-\frac{t}{nX_{n:n}}$
$\bar{F}(t)$	$\hat{\bar{F}}(t) = 1-\frac{t}{X_{n:n}}\left(\frac{n-1}{n}\right)$	$\left(\frac{n-1}{n^2}\right)\frac{t}{X_{n:n}}$
	$\tilde{\bar{F}}(t) = 1-\frac{t}{X_{n:n}}$	$\frac{t}{nX_{n:n}}$
$\boldsymbol{\lambda}(t)$	$\hat{\boldsymbol{\lambda}}(t) = \frac{1}{X_{n:n}-t}\left\{1-\frac{X_{n:n}}{n(X_{n:n}-t)}\right\}$	$-\frac{X_{n:n}(t+nt+X_{n:n}-nX_{n:n})}{n^2(t-X_{n:n})^3}$
	$\tilde{\boldsymbol{\lambda}}(t) = \frac{1}{X_{n:n}-t}$	$-\frac{X_{n:n}}{n(X_{n:n}-t)^2}$

substituting $T(X_{n:n})$ as appeared in (14) for (13).

Examples: In context of remark
The pdf $f(x, \theta)$ defined in (1) can be characterized through non constant function such as

$$g_i(\theta) = \begin{cases} \frac{\theta}{2}; for\ i = 1, Mean, \\ \frac{\theta^r}{r+1}; for\ i = 2,\ \text{r}^{\text{th}} \text{ raw moment}, \\ e^{\theta}; for\ i = 3, \\ e^{-\theta}; for\ i = 4, \\ \theta p; for\ i = 5,\ \text{p}^{\text{th}} \text{ quantile}, \\ \frac{t}{\theta}; for\ i = 6,\ \text{distribution function at t}, \\ 1 - \frac{t}{\theta}; for\ i = 7,\ \text{Relibility at t}, \\ 1 - \frac{t}{\theta}; for\ i = 8,\ \text{Hazard function}, \end{cases}$$

and by using

$$[\phi_i(X) - g_i(X)] = \begin{cases} \frac{x}{2}; for\ i = 1, Mean, \\ \frac{r}{r+1}x^r; for\ i = 2,\ \text{r}^{\text{th}} \text{ raw moment}, \\ xe^{x}; for\ i = 3, \\ -xe^{-x}; for\ i = 4, \\ xp; for\ i = 5,\ \text{pth quantile}, \\ -\frac{t}{x}; for\ i = 6,\ \text{distribution Function at t}, \\ \frac{t}{x}; for\ i = 7,\ \text{Relibility Function at t}, \\ -\frac{x}{(x-t)^2}; for\ i = 8,\ \text{Hazard Function}, \end{cases}$$

and defining $M(X)$ given in (12) and substituting $T(X)$ as appeared in (14) for (13).

Conclusion

To characterize pdf given in (12) one needs any arbitrary non constant function only.

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