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Finite Volume Numerical Grid Technique for Solving One and Two Dimensional Heat Flow Problems

J.S.V.R. Krishna Prasad and Patil Parag Vijay
Department of Mathematics, M. J. College, Jalgaon, 425 001, Maharashtra INDIA

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Abstract

In this paper Finite Volume numerical technique has been used to solve one and two dimensional Steady state heat flow problems with Dirichlet boundary conditions and mixed boundary conditions, respectively. We explained step by step numerical solution procedures with the help of Microsoft excel and TDMA line-by-line solver for the algebraic equations. Finally the numerical solutions obtained by Finite Volume techniques are compared with exact solution to check the accuracy of the developed scheme

Keywords: Finite volume technique, steady state heat flow equation, dirichlet boundary conditions, mixed boundary conditions, TDMA Solver.

Introduction

In the last few decades, revolution in the computer technology has led to development of numerous computational grid techniques for solving many engineering problems[1-3]. As mathematical modelling became an integral part of analysis of engineering problems, a variety of numerical grid techniques have been developed. A commonly used numerical technique is the finite difference method (FDM), described in references[4-6]. The another numerical technique called the finite element method (FEM) developed originally for the solution of structural problem, has been applied to the solution of heat conduction problems and other details about this technique can be seen in the papers[4-9]. The next popular numerical technique is finite volume method (FVM) was originally developed as a special finite difference formulation; for more detailed the reader may consult[10]. Each of these methods has its own merits and demerits depending on the problem to be solved. Out of the available numerical gird techniques, the finite volume technique is one of the most flexible and versatile technique for solving the problems in computational fluid dynamics.

The remainder of the paper is organised as follows. In Section 2, a short review of finite volume techniques with the help of TDMA (Tri-Diagonal Matrix Algorithm) solver is given. In Section 3, formulation one and two dimensional heat flow problems with Dirichlet and Mixed boundary conditions. Also, we explained step by step numerical solution procedures with the help of Microsoft excel. In Section 4, the numerical solutions obtained by this technique are compared with exact solution. Finally, Section 5 concludes the paper.

Finite Volume Grid Technique: The Finite Volume Method is an increasing popular numerical technique for the approximate solution of partial differential equations. For more detailed the reader may consult[10]. The Finite Volume analysis involves three basic steps. i. The problem domain is defined and divided the solution domain into discrete control volume. Let us place a numbers of nodal points in the given space and domain is divided in such way that, each node is surrounded by the control volume or grid and the physical boundaries coincide with the control volume boundaries. ii. The integration of the governing equation over the control volume to yield a discretised equation at its nodal point. iii. Solve the set of discretised equations using TDMA solver.

Finite Volume Discretizations: The General form of discretised equations for one and two dimensional steady state heat flow problems are given by equation (1).

$$a_p\theta_p = \sum a_i\theta_i + S_\theta \quad (1)$$

$$a_p = \sum a_i - S_p \quad (2)$$

$$a_i = \frac{kA}{\Delta} \quad (3)$$

Where a_i are the neighbouring coefficients a_W, a_E and a_W, a_E, a_N, a_S in one and two dimensional respectively.θ_i are the values of the function θ at the neighbouring nodes. S_θ and S_p are the values obtained from the linear source term $S_\theta + S_p\theta_p$ which is the function of the dependent variable. Note that, to obtain the values S_θ and S_p from the linear source term $S_\theta + S_p\theta_p$ with boundary B.

For Fixed value θ_B ,

$$S_\theta = \frac{2kA}{\Delta}\theta_B \text{ and } S_p = -\frac{2kA}{\Delta}$$

For Fixed Flux q,

$$S_\theta = q \times A \text{ and } S_p = 0$$

Tdma (Tri-Diagonal Matrix Algorithm): The tri diagonal matrix algorithm (TDMA), also known also Thomas algorithm, is a simplified form of Gaussian elimination that can be used to solve tri diagonal system of equations

$$-a_i\theta_{i-1} + b_i\theta_i - c_i\theta_{i+1} = d_i \quad (4)$$
$$i = 1, - - - -, n$$

The TDMA is based on the Gaussian elimination procedure and consist of two parts - a forward elimination phase and a backward substitution phase. The TDMA is actually a direct method for one dimensional situation, but it can be applied iteratively in a line-by-line fashion, to solve multidimensional problems and is widely used in CFD programs. Let us consider the system for $i = 1, - - - -, n$ and we use the general form of the TDMA solver is given by

$$\theta_i = A_i\theta_{i+1} + B_i \quad (5)$$

Where

$$A_i = \frac{c_i}{b_i - a_iA_{i-1}} \text{ and } B_i = \frac{a_iB_{i-1} + d_i}{b_i - a_iA_{i-1}}$$

To solve the above system TDMA is applied for one dimensional problem, the discretised equation is re-arranged in the form

$$-a_W\theta_W + a_P\theta_P - a_E\theta_E = S_\theta \quad (6)$$

To solve the above system TDMA is applied along the north-south lines for two dimensional problems, the discretised equation is re-arranged in the form

$$-a_S\theta_S + a_P\theta_P - a_N\theta_N = a_W\theta_W + a_E\theta_E + S_\theta \quad (7)$$

Problem Formulation

Problem I: Consider one dimensional steady state heat flow in the iron rod with Dirichlet boundary conditions, the mathematical formulation of this problem is given by

$$\frac{d}{dx}\left(\frac{d\theta}{dx}\right) - N^2\theta(x) = 0 \;\; in\; 0 < x < L \quad (8)$$

Subject to the Dirichlet boundary conditions
$\theta(x) = \theta_0 \quad at\; x = 0$
$\theta(x) = 0 \quad at\; x = L$
as shown in figure-1.

Where $\theta(x) = T(x) - T_\infty$
$\theta_0 = T_0 - T_\infty$

$$N^2 = \frac{Ph}{Ak} = \frac{\pi Dh}{(\pi/4)D^2k} = \frac{4h}{Dk}$$

The Exact solution of this problem is given by ,

$$\theta(x) = \theta_0\frac{Sinh\, N(L-x)}{Sinh\, NL} \quad (9)$$

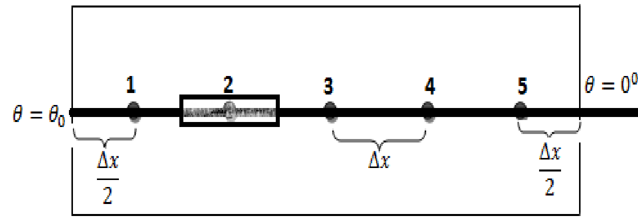

Figure-1
Solution region with Dirichlet boundary conditions

Let us introduce,
The thermal conductivity $k = 50\; W/m.^0C$
The length of the rod $L = 0.1\; m$
The thickness of the rod $D = 0.02\; m$
The heat transfer coefficient $h = 200\; W/m^2.^0C$
The grid size $\Delta x = 0.02\; m$
$\theta_0 = 200^0C$ and the ambient temperature $T_\infty = 0^0C$

The coefficients and the source term of the discretisation equation for all nodes are summarised in Table-1 .The numerical solution of the discretised equations system is calculated using TDMA with the help of Microsoft excel as shown in Table-2.

Table-1
The coefficients and source term for all nodes

Node	a_W	a_P	a_E	S_θ
1	0	182	50	20000
2	50	132	50	0
3	50	132	50	0
4	50	132	50	0
5	50	182	0	0

Table-2
The Numerical Solution using TDMA

Node	d_i	A_i	B_i	θ_i
1	20000	0.2747	109.8901	125.6610
2	0	0.4228	46.4598	57.4061
3	0	0.4510	20.9541	25.8911
4	0	0.4568	9.5725	10.9463
5	0	0.0000	3.0072	3.0072

Problem II: Consider two dimensional steady state heat transfers in the plate with mixed boundary conditions; the mathematical formulation of this problem is given by

$$\frac{\partial}{\partial x}\left(k\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial\theta}{\partial y}\right) = 0 \;\; in\; 0 \le x, y \le 1 \quad (10)$$

Subject to the mixed boundary conditions
$\theta = 1 + 2y\; at\; x = 0, 0 \le y \le 1$
$\theta = 2 + 2y\; at\; x = 1, 0 \le y \le 1$
$q = -2\; at\; y = 0, 0 \le x \le 1$
$q = 2\; at\; y = 1, 0 \le x \le 1$
as shown in figure -2.

The Exact solution of this problem is given by

$$\theta(x,y) = 1 + x + 2y \tag{11}$$

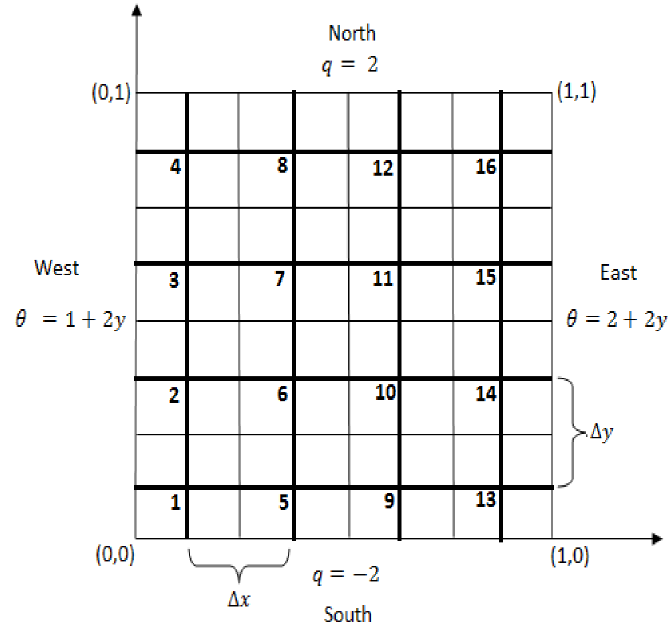

Figure-2
Solution region with mixed boundary condition

Let us introduce,
The thermal conductivity $k = 1000\ W/m/k$
The thickness of the plate $D = 0.25$
The grid size $\Delta x = \Delta y = 0.25$
The Area $A = 0.25 \times 0.25\ m^2$
The coefficients and the source term of the discretisation equation for all points are summarised in Table-3.

Let us apply TDMA using Microsoft excel along north-south lines, sweeping from west to east. For convenience the line in Figure 2 containing points 1 to 4 referred to as line 1, points 5 to 8 as line 2, points 9 to 12 as line 3 and the one with points 13 to 16 as line 4. At the end of the first iteration we have the values shown in Table -4 for the entire field.

The entire procedure is now repeated until a converged solution is obtained. In this case after 7 iterations we obtained the converged solution as shown in following Table 5.

Results and Discussion

All the numerical calculations were done with control volume grids for one and two dimensional heat flow problems respectively using Microsoft excel. Finally the numerical solutions obtained by Finite Volume techniques are compared with exact solution to check the accuracy of the developed scheme as shown in table 6 and 7.

Table-3
The coefficients and source term for all nodes

Node	a_N	a_P	a_S	a_W	a_E	S_θ
1	250	1000	0	0	250	624.87
2	250	1250	250	0	250	875
3	250	1250	250	0	250	1125
4	0	1000	250	0	250	1375.12
5	250	750	0	250	250	-0.125
6	250	1000	250	250	250	0
7	250	1000	250	250	250	0
8	0	750	250	250	250	0.125
9	250	750	0	250	250	-0.125
10	250	1000	250	250	250	0
11	250	1000	250	250	250	0
12	0	750	250	250	250	0.125
13	250	1000	0	250	0	1124.87
14	250	1250	250	250	0	1375
15	250	1250	250	250	0	1625
16	0	1000	250	250	0	1875.13

Table-4
The Numerical Solution after first iteration

Node	θ_W	θ_E	d_i	A_i	B_i	θ_i
--	--	--	--	0.0000	0.0000	--
1	0.0000	0	624.87	0.2500	0.6249	0.9202
2	0.0000	0	875.00	0.2105	0.8684	1.1811
3	0.0000	0	1125.00	0.2088	1.1209	1.4855
4	0.0000	0	1375.12	0.0000	1.7465	1.7465
5	0.9202	0	229.91	0.3333	0.3066	0.5081
6	1.1811	0	295.28	0.2727	0.4057	0.6045
7	1.4855	0	371.38	0.2683	0.5074	0.7288
8	1.7465	0	436.75	0.0000	0.8253	0.8253
9	0.5081	0	126.89	0.3333	0.1692	0.2721
10	0.6045	0	151.13	0.2727	0.211	0.3086
11	0.7288	0	182.21	0.2683	0.2522	0.358
12	0.8253	0	206.44	0.0000	0.3946	0.3946
13	0.2721	0	1192.89	0.2500	1.1929	1.6809
14	0.3086	0	1452.16	0.2105	1.474	1.952
15	0.3580	0	1714.50	0.2088	1.7397	2.2703
16	0.9202	0	1973.77	0.0000	2.5413	2.5413

Table-5
The Numerical solution after 7th Iterations

Node	θ_W	θ_E	d_i	A_i	B_i	θ_i
				0.000	0.000	
1	0.000	1.945	1111.17	0.250	1.111	1.586
2	0.000	2.132	1408.04	0.211	1.420	1.898
3	0.000	2.381	1720.31	0.209	1.733	2.273
4	0.000	2.568	2017.17	0.000	2.585	2.585
5	1.586	2.227	952.97	0.333	1.271	2.000
6	1.898	2.414	1078.04	0.273	1.523	2.187
7	2.273	2.664	1234.13	0.268	1.733	2.437
8	2.585	2.851	1359.20	0.000	2.625	2.625
9	2.000	2.346	1086.36	0.333	1.449	2.267
10	2.187	2.659	1211.51	0.273	1.717	2.454
11	2.437	3.034	1367.71	0.268	1.928	2.704
12	2.625	3.346	1492.86	0.000	2.892	2.892
13	2.267	0.000	1691.51	0.250	1.692	2.360
14	2.454	0.000	1988.55	0.211	2.031	2.672
15	2.704	0.000	2301.06	0.209	2.346	3.047
16	2.892	0.000	2598.10	0.000	3.360	3.360

Table-6
A Comparison between Numerical solutions with Exact for Problem I

Node	FVM	Exact	Error
1	125.661	134.0089	8.3479
2	57.4061	60.0362	2.6301
3	25.8911	26.5802	0.6892
4	10.9463	11.0624	0.1161
5	3.0072	3.0103	0.0031

Table-7
A Comparison between Numerical Solutions with Exact for Problem II

Node	FVM	Exact	Error
1	1.5857	1.3750	0.2107
2	1.8982	1.8750	0.0232
3	2.2730	2.3750	0.1020
4	2.5854	2.8750	0.2896
5	1.9997	1.6250	0.3747
6	2.1872	2.1250	0.0622
7	2.4371	2.6250	0.1879
8	2.6246	3.1250	0.5004
9	2.2666	1.8750	0.3916
10	2.4542	2.3750	0.0792
11	2.7042	2.8750	0.1708
12	2.8919	3.3750	0.4831
13	2.3596	2.1250	0.2346
14	2.6722	2.6250	0.0472
15	3.0473	3.1250	0.0777
16	3.3599	3.6250	0.2651

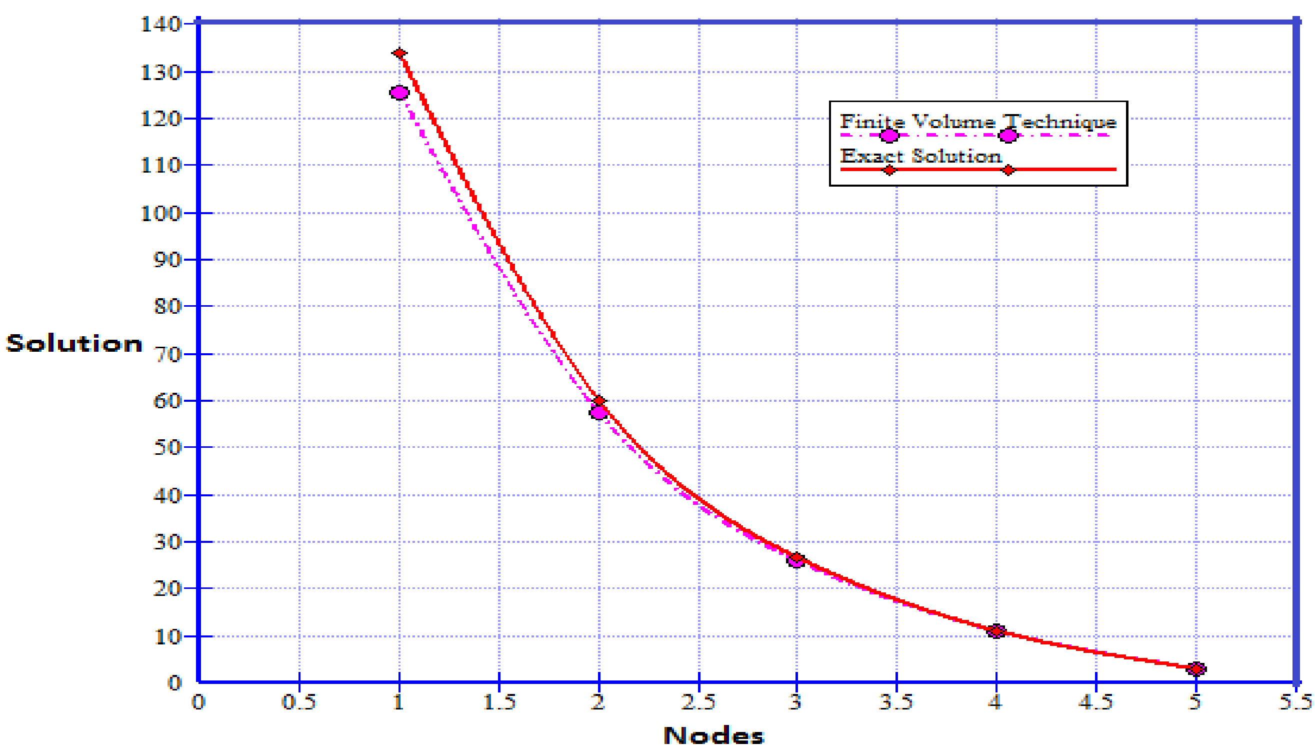

Figure-3
A comparison between Finite Volume Numerical solution with Exact Solution for Problem I

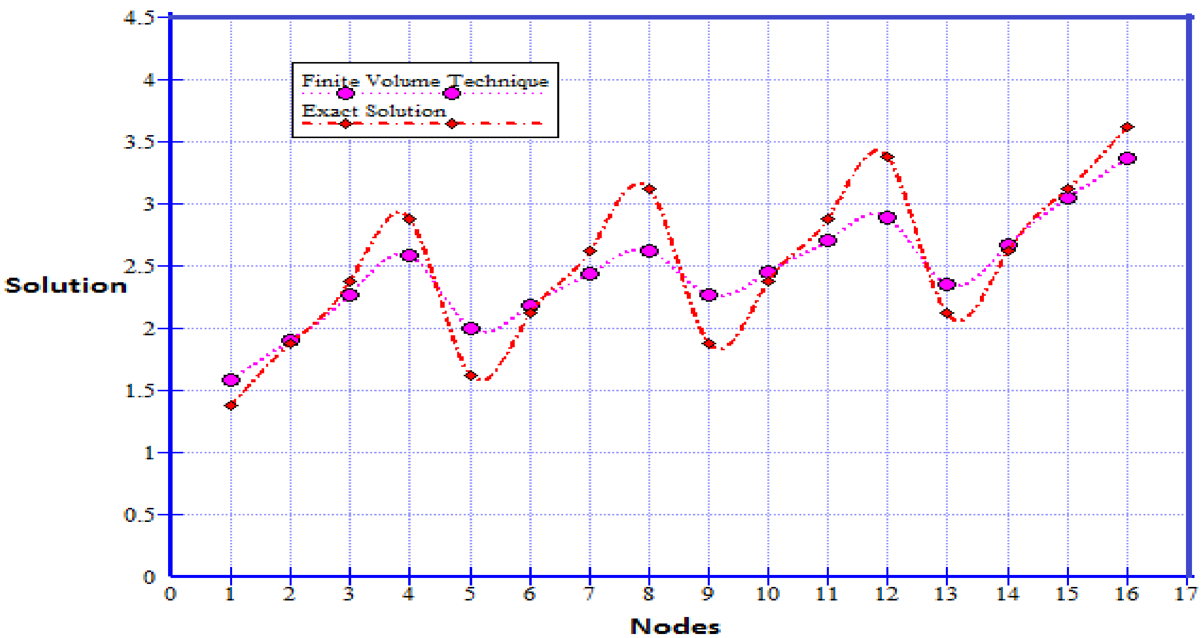

Figure-4
A comparison between Finite Volume Numerical Solution with Exact Solution for Problem II

Conclusion

In this work, we have studied finite volume numerical grid technique for steady state heat flow problems and obtained the numerical solution of the one and two dimensional heat flow equation with Dirichlet boundary conditions and mixed boundary conditions, respectively. We have used TDMA solver for solving algebraic equations and the results obtained by this technique are all in good agreement with the exact solutions under study. Moreover this technique is efficient, reliable, accurate and easier to implement in Microsoft excel as compared to the other costly techniques.

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