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Inverse Maxwell Distribution as a Survival Model, Genesis and Parameter Estimation

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Abstract

If a random variable follows a particular distribution then the distribution of the inverse of that random variable is called inverted distribution. Some authors have discussed both continuous and discrete inverted distribution and its applications to various disciplines eg. In social sciences to geological, engineering, environmental and medical sciences etc. In this paper we have derived the probability density function of Inverse Maxwell distribution and studied its properties and its suitability as a survival model has been discussed by obtaining its survival and hazard functions, these are plotted on a graph paper and their properties have been discussed. The maximum likelihood estimator, an moment equation estimator of the parameter have been obtained

Keywords: Maxwell distribution, inverse distribution, moments, maximum likelihood estimation**,** moment equation estimation of parameter, survival and hazard functions etc.

Introduction

Let X be a random variable having the Maxwell distribution, with pdf given by:

$$f(x;\theta) = \frac{4}{\sqrt{\pi}}\frac{x^2}{\theta^{\frac{3}{2}}}e^{\frac{-x^2}{\theta}} \quad x > 0, \theta > 0 \qquad (1)$$

Where θ is the scale parameter. The Maxwell distribution, the raw moments are given by

$$\mu_r' = \frac{2}{\sqrt{\pi}}\theta^{\frac{r}{2}}\Gamma(\frac{r+3}{2}) \qquad (2)$$

The mean and variance are given as

$$\mu_1' = E(x) = 2\sqrt{\frac{\theta}{\pi}}\,; \qquad (3)$$

and $\mu_1' = V(x) = \frac{\theta(3\pi-8)}{2\pi}$ (4)

If X is a random variable then Y = 1/X, is described as inverted random variable. There are many inverted distributions discussed in literature. Stephan[1] is one of the earlier authors who discuss negative moments of binomial and hyper geometric distributions. Grab and Savage[2] have constructed the tables for negative moments of the binomial and Poisson distributions. Mendenhall and Lehman[3], Govendarajulu[4,5], Tiku[6], Vijsokousku[7], Stancu[8], Skibusky[9], Kabe[10] Shahnbhag and Busawa[11], Chao and Strawderman[12], Gupta[13,14], Lepage [15], Kumar and Cousul [16], Cressie[17]et al, Cressie[18]et al, Ahmad and Sheikh[19,19,20,21] Jones[22], are some of the early authors who have discussed various aspects of inverted distributions or negative moments. More recently, Roohi[23], Jones and Zhiglijavsky[24], Rempala[25] and Ahmad and Roohi[26,27,28,29], have also discussed both continuous and discrete inverted distribution.

In this paper we have obtained the p.d.f. of inverse Maxwell distribution and obtained its mean, variance, harmonic mean and mode. The suitability of Inverse Maxwell distribution as a survival model has been discussed by obtaining the hazard and survival functions.

Inverse Maxwell Distribution

If X has a Maxwell distribution then the random variable Y= $\frac{1}{X}$ is said to follow inverse Maxwell distribution. The pdf of inverse Maxwell distribution may be obtained using the transformation technique. We get

$$f(y;\theta) = f_X(1/y;\theta)\left|\frac{d}{dy}y^{-1}\right| \qquad (5)$$

where $f_X(.;\theta)$ is given by (1). Now (5) becomes

$$f(y;\theta) = \frac{4}{\sqrt{\pi}\theta^{\frac{3}{2}}}\frac{1}{y^4}.e^{-\frac{1}{\theta y^2}} \quad y > 0, \theta > 0 \qquad (6)$$

Here we get, $\int_0^\infty f(y;\theta) = 1.$

Moments: The inverse Maxwell distribution, the rth raw moments are given by

$$E(Y^r) = \mu_r' = \int_0^\infty y^r f(y)dy$$

$$= \frac{2}{\sqrt{\pi}\theta^{\frac{r}{2}}}\Gamma(\frac{-r+3}{2}) \qquad (7)$$

The mean, variance, harmonic mean and mode are obtained as

Mean $=\int_0^\infty y\,f(y)dy$

$$\mu_1' = \frac{2}{\sqrt{\pi\theta}} \qquad (8)$$

$$\mu_2' = \int_0^\infty y^2 f(y)dy = \frac{2}{\theta}$$

Variance = $\mu_2 = \frac{2(\pi-2)}{\theta\pi}$ (9)

$\frac{1}{H} = \int_0^\infty \frac{1}{y} f(y)dy$

Harmonic Mean

$\frac{1}{H} = \int_0^\infty \frac{1}{y} f(y)dy$

$= H = \frac{1}{2}\sqrt{\frac{\pi}{\theta}}$ (10)

The most likely value, that is, *y*i that has the highest probability *pi*, or the *y* at which pdf is maximum, is called mode of *y*.

Mode

$\frac{d}{dy} f(y)dy = 0$

$M_o = \frac{1}{2}\sqrt{\frac{\pi}{\theta}}$ (11)

It is clear from figure1 that the density function of the Inverse Maxwell distribution takes different shapes for different values of the parameter θ. The curve is positively skewed and has a long right tail. It is unimodal and its mode shifts in the right side and the tail increases and the curve tends to be symmetrical and flatter as θ decreases.

Cumulative distribution function: The cumulative distribution function of a real-valued random variable *Y* is the function given by

$F_Y(y) = P(Y \leq y)$

where the right-hand side represents the probability that the random variable Y takes on a value less than or equal to *y*. Now, the cumulative distribution function of Inverse Maxwell distribution is defined as

$F(t) = \int_0^t f(y;\ \theta)\, dy$ (12)

$= \int_0^t \frac{4}{\sqrt{\pi}\theta^{\frac{3}{2}}} \frac{1}{y^4} . e^{-\frac{1}{\theta y^2}} dy$

Put $\frac{1}{\theta y^2} = u$; $\frac{-2dy}{\theta y^3} = du$

Limits y = 0 ; u = ∞ ; y = t ; u = $\frac{1}{\theta t^2}$ and we get

$F(t) = \frac{2}{\sqrt{\pi}} \int_{\frac{1}{\theta t^2}}^{\infty} u^{\frac{3}{2}-1} e^{-u} du$

$= 1 - \frac{2}{\sqrt{\pi}} \gamma(\frac{3}{2}, \frac{1}{\theta t^2})$ (12)

The cdf of inverse Maxwell distribution is lower incomplete gamma function depending on θ. It is 'S' shaped. And its shape changes in accordance with θ.

Survival function: Let T be a continuous random variable with probability density function (p.d.f.) f(t) and cumulative distribution function (c.d.f.) F(t) = Pr(T ≤ t), giving the probability that the event has occurred by duration t. then the survival function is given as

$S(t) = Pr\{T > t\} = 1 - F(t)$ (14)

$= 1 - [1 - \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}, \frac{1}{\theta t^2})] = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta t^2}\right)$ (15)

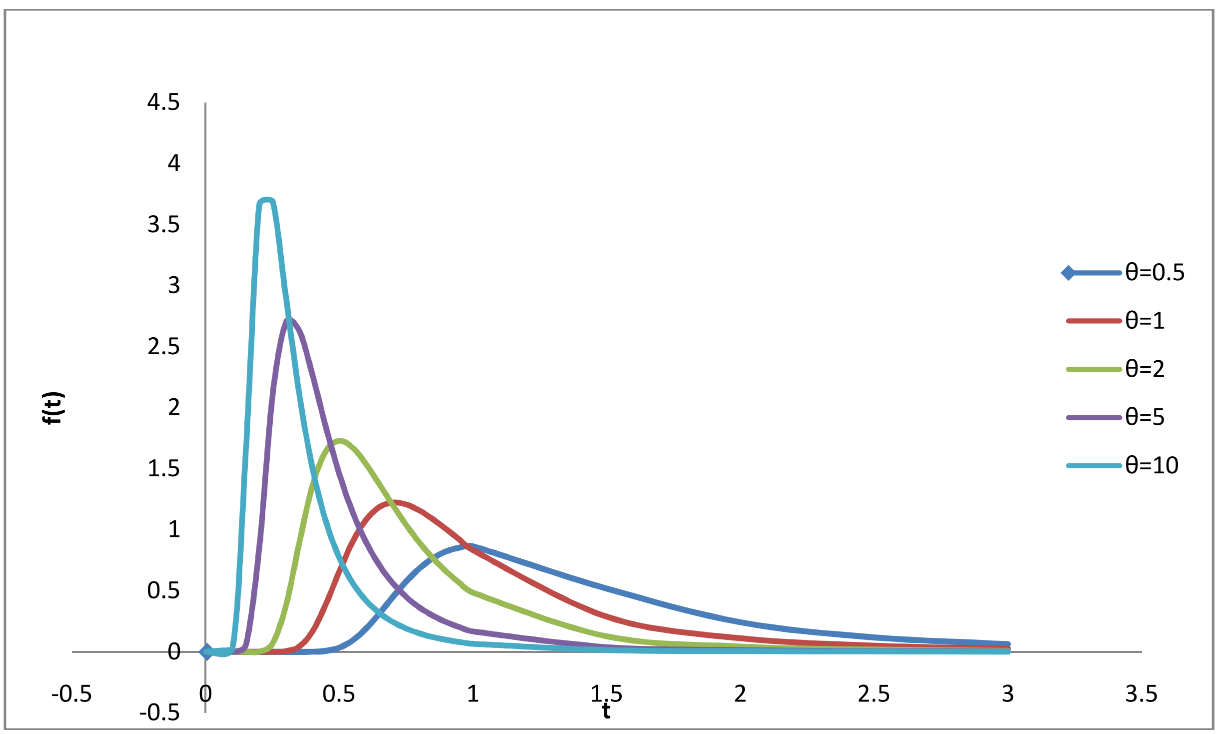

Figure-1
pdf of Inverse Maxwell distribution

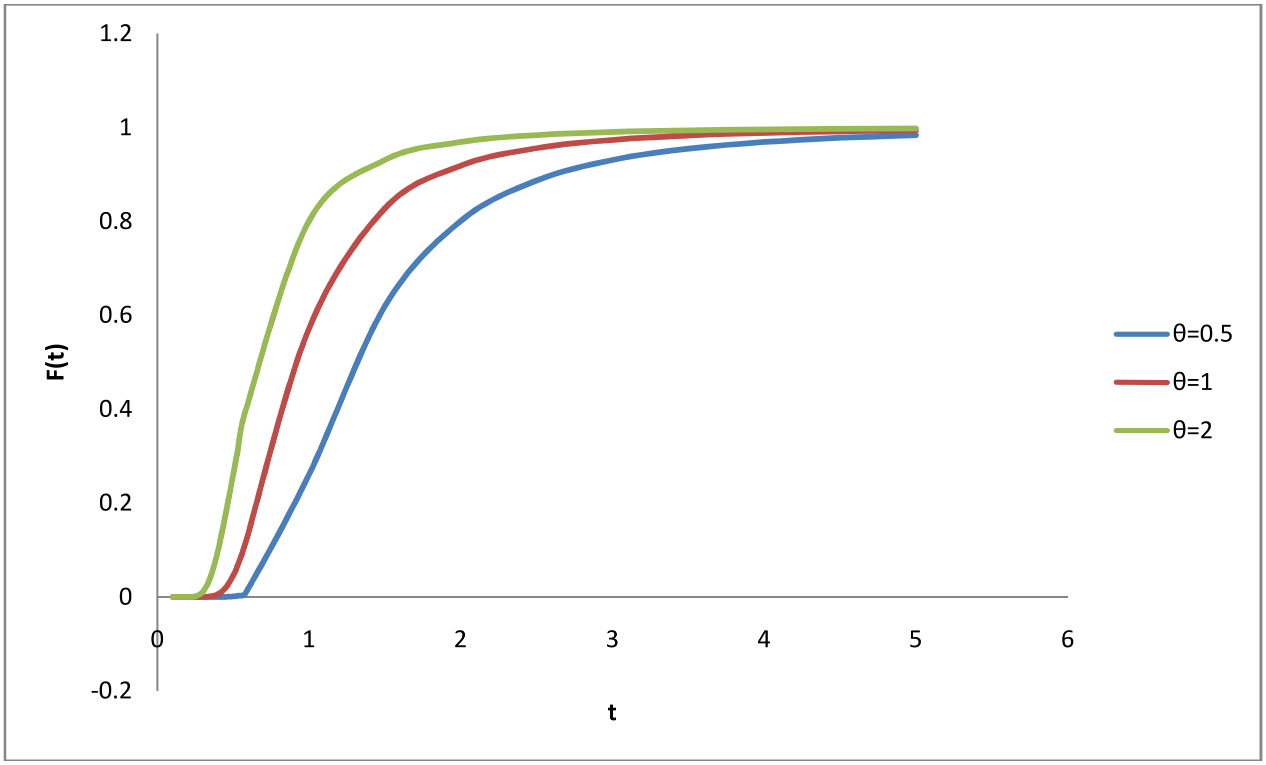

Figure-2
cdf of inverse Maxwell distribution

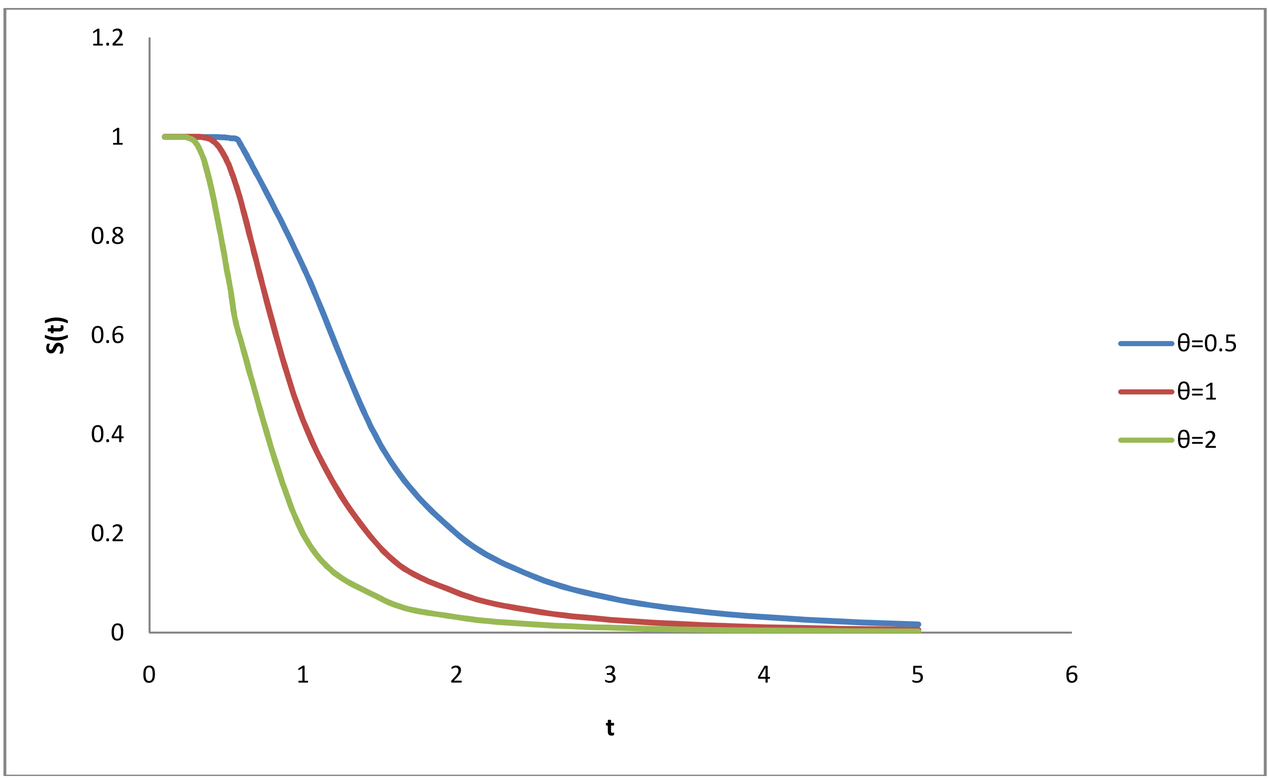

Figure-3
Survival function of inverse Maxwell distribution

For a survival function, the t value on the graph starts at one and monotonically decreases to zero. The inverse Maxwell distribution's survival function is shown in the figure below.

Hazard function: the hazard function, or instantaneous failure rate λ(t) function is defined by,

$$\lambda\,(t) = \lim_{dt\to 0}\frac{\Pr\{t<T\le t+dt|T>t\}}{dt}$$

The conditional probability in the numerator may be written as the ratio of the joint probability that T is in the interval (t; t + dt) and T > t (which is, of course, the same as the probability that t is in the interval), to the probability of the condition T > t. The former may be written as f(t)dt for small dt, while the latter is S(t) by definition. Dividing by dt and passing to the limit gives the useful result.The hazard function of IMD is define as

$$\lambda\,(t) = \frac{f(t)}{S(t)} \qquad (16)$$

where f(t) is probability density function and S(t), survival function of inverse Maxwell distribution respectively. We have,

$$\lambda\,(t) = \frac{\frac{4}{\sqrt{\pi}\theta^{\frac{3}{2}}}\frac{1}{t^4}.e^{-\frac{1}{\theta t^2}}}{\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{1}{\theta t^2})}\;;t>0,\theta>0$$

$$=\frac{1}{\gamma(\frac{3}{2},\frac{1}{\theta t^2})}.\frac{2e^{-\frac{1}{\theta t^2}}}{t^4\theta^{\frac{3}{2}}}\;;t>0,\theta>0 \qquad (17)$$

The hazard function of IMD increases initially, then decreases and eventually approaches zero. This means that items with a inverse Maxwell distribution have a higher chance of failing as they age for some period of time, but after survival to a specific age, the probability of failure decreases as time increases. The inverse Maxwell distribution's hazard function is shown in the figure below. For θ = 0.5,1.00,2.00.for smaller value of θ the hazard function are steeper.

Estimation of Parameters

Moment equation estimation: In this method of moments replacing the population mean are equated with the method of moments estimation the sample moments are equated with the corresponding population moments and solved for the parameter(s). Thus the corresponding sample mean and variance respectively,

$$\mu_1' = \bar{y} \qquad (18)$$

Putting the value of μ_1' in equation (18),we have

$$\frac{2}{\sqrt{\pi\theta}} = \bar{y}$$

Squaring both the sides, we get

$$\frac{4}{\pi\theta} = \bar{y}^2$$

From the above equation, we have the sample mean of θ is

$$\hat{\theta} = \frac{4}{\pi\bar{y}^2} \qquad (19)$$

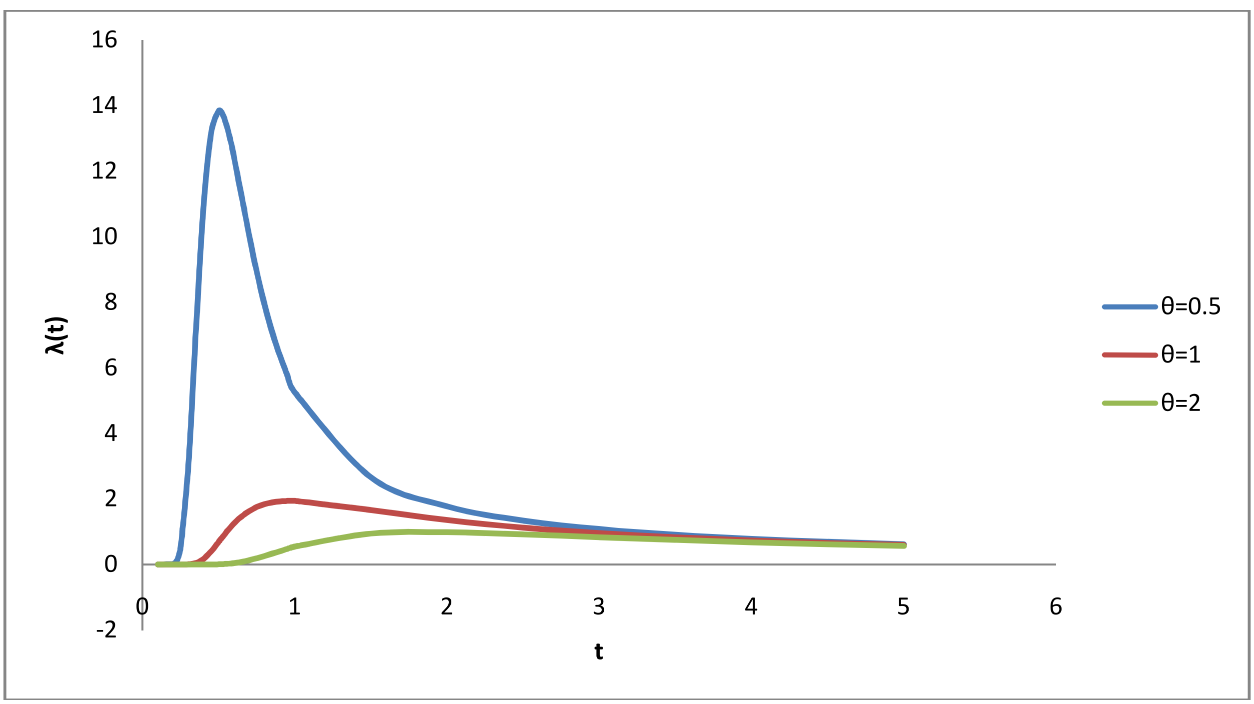

Figure-4
hazard function of inverse Maxwell distribution for different value of θ.

Maximum Likelihood Estimator (MLE)

Case (i) when r$< n$: Let n items are put to test for their lifetimes, their failure (life) times are recorded and the experiment is terminated as soon as r≤n (preassigned) items fail. Let the reordered lives are as $y_1 < y_2 < \ldots\ldots\ldots < y_r$ (r≤ n). now likelihood function is

$$L \propto \prod_{i=1}^{r} f(y;\theta)^{r}\,[1-F(y_r;\theta)]^{n-r} \quad (20)$$

$$L = k\left(\frac{4}{\sqrt{\pi}}\right)^{r}\frac{1}{\theta^{\frac{3r}{2}}}\left(\prod_{i=1}^{n}\frac{1}{y_i^4}\right).e^{-\left(\sum_{i=1}^{n}\frac{1}{y_i^2}+(n-r)\frac{1}{y_r^2}\right)\left(\frac{1}{\theta}\right)}\left[\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_r^2}\right)\right]^{(n-r)} \quad (21)$$

Taking log on both side, we have

$$\text{LogL}= r\,[\log 4 - \log\sqrt{\pi}] + \log\left(\prod_{i=1}^{r}\frac{1}{y_i^4}\right) + r[\log 1-\frac{3}{2}\log\theta] - \left(\sum_{i=1}^{n}\frac{1}{y_i^2}+(n-r)\frac{1}{y_r^2}\right)\left(\frac{1}{\theta}\right) + (n-r)\log\left[\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_r^2}\right)\right] \quad (22)$$

Differentiating equation (22) w.r.t. θ and setting the results equal to zero, we have

$$\frac{d\log L}{d\theta} = \frac{-3n}{2\theta} + \left(\sum_{i=1}^{n}\frac{1}{y_i^2}+(n-r)\frac{1}{y_r^2}\right)\frac{1}{\theta^2} + \frac{\frac{2}{\sqrt{\pi}}(n-r)\frac{d}{d\theta}\left[\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_r^2}\right)\right]}{\left[\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_r^2}\right)\right]}$$

$$\frac{d\log L}{d\theta} = \frac{-3n}{2\theta} + \left(\sum_{i=1}^{n}\frac{1}{y_i^2}+(n-r)\frac{1}{y_r^2}\right)\frac{1}{\theta^2} - \frac{(n-r)y_r\theta^{\frac{3}{2}}e^{-\frac{1}{\theta y_r^2}}}{\gamma\left(\frac{3}{2},\frac{1}{\theta y_r^2}\right)}$$

$$\frac{d\log L}{d\theta} = 0$$

$$\frac{-3n}{2\theta} + \left(\sum_{i=1}^{n}\frac{1}{y_i^2}+(n-r)\frac{1}{y_r^2}\right)\frac{1}{\theta^2} - \frac{(n-r)y_r\theta^{\frac{3}{2}}e^{-\frac{1}{\theta y_r^2}}}{\gamma\left(\frac{3}{2},\frac{1}{\theta y_r^2}\right)} = 0 \quad (23)$$

It is a non-linear equation in θ. which can not be solved directly. Its solution may be obtained either by Newton-Raphson method or method of scoring. The maximum likelihood equations involve the parameters in limit of integral in the incomplete gamma function. So we cannot get simplified solution of the above equation.

Case (ii) when r = n: The estimations of the parameter of inverse Maxwell distribution are obtained by the method of MLE using equation (6), the maximum log likelihood function of the IMD as follow:

$$L = \prod_{i=1}^{n} f(y;\theta) \quad (24)$$

$$= \prod_{i=1}^{n}\frac{4}{\sqrt{\pi}\theta^{\frac{3}{2}}}\,\frac{1}{y^4}.e^{-\frac{1}{\theta y^2}}$$

$$= \left(\frac{4}{\sqrt{\pi}}\right)^{n}\left(\frac{1}{\theta^{\frac{3}{2}}}\right)^{n}\left(\prod_{i=1}^{n}\frac{1}{y_i^4}\right).e^{-\left(\sum_{i=1}^{n}\frac{1}{y_i^2}\right)\left(\frac{1}{\theta}\right)}$$

Now the log likelihood function for n observation of Y is given by

$$\text{Log L} = n\,[\log 4 - \log\sqrt{\pi}] + \log\left(\prod_{i=1}^{n}\frac{1}{y_i^4}\right) + n\,[\log 1-\frac{3}{2}\log\theta] - \left(\sum_{i=1}^{n}\frac{1}{y_i^2}\right)\left(\frac{1}{\theta}\right) \quad (25)$$

Differentiating equation (25) w.r.t. θ and setting the results equal to zero, we have

$$\frac{d\log L}{d\theta} = \frac{-3n}{2\theta} + \left(\sum_{i=1}^{n}\frac{1}{y_i^2}\right)\frac{1}{\theta^2} \quad (26)$$

now, $\frac{d\log L}{d\theta} = 0$

leads to

$$\hat{\theta} = \left(\frac{2}{3n}\right)\left(\sum_{i=1}^{n}\frac{1}{y_i^2}\right) \quad (27)$$

As the MLE of θ

Conclusion

In this paper we have obtained the p.d.f. of inverse Maxwell distribution by the help of transformation technique an also plotted pdf in figure.1, and studied their properties. Its suitability as a survival model has been discussed by obtaining its survival and hazard functions. These are plotted in a figures-3 and 4 respectively, and their properties have been discussed. The maximum likelihood estimator of IMD when r $< n$, a bit difficulty. Since the maximum likelihood equation involve the parameter in limit of integral, so we cannot get simplified solution of the equation. Now when r = n is, the MLE of θ is

$$\hat{\theta} = \left(\frac{2}{3n}\right)\left(\sum_{i=1}^{n}\frac{1}{y_i^2}\right)$$

and moment equation estimator of θ have been obtained as

$\hat{\theta} = \frac{4}{\pi\bar{y}^2}$;

At suitability as a survival model has been proposed by obtaining the hazard function. the IMD may be a good survival model when the data conforms with its hazard function and survival function. Suitable example may be obtained an literature.

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