Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. **2(7),** 17-22, July **(2014)** *Res. J. Mathematical and Statistical Sci.*

A Stability Analysis on Models of Cooperative and Competitive Species

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Available online at: www.isca.in, www.isca.me
Received 27th May 2014, revised 11th June 2014, accepted 10th July 2014

Abstract

This paper presents a stability analysis on generalised mathematical models for cooperative and competitive species. For each system, we determined all the relevant equilibrium points and analysed the behavior of solutions whose initial conditions satisfy either $x_1 = 0$ or $x_2 = 0$. The curves in the phase plane along which the vector field is either horizontal or vertical were determined. For each of the systems, we described all possible population scenarios using the phase potraits. The cooperative system was found to be stable at one of the two equilibrium points presents and unstable (Saddle) at the other. Four equilibrium points existed for the competitive species model for which the system is stable at one point and locally asymptotically stable at the other three points. The asymptotical stability is based on the inhibition and the coexistence factors between the two competing species.

Keywords: Cooperative species, competitive species, stability, inhibition, coexistence.

Introduction

There are two types of predator-prey system which are competitive and cooperative systems. Competitive is the one in which both species are harmed by each other by interaction for example, cars and pedestrians and cooperative is the one in which both species benefit by their interactions for example, bees and flowers. Some series of research has been done in this area. Srilatha in 2012[1] investigated mathematical model of a four species Syn-Ecological system (The Coexistent State). The constituent of the model equations is the set of four first order nonlinear ordinary differential coupled equations with sixteen equilibrium points. The criteria for the asymptotic stability (Co-existent State) of one of the sixteen equilibrium points was only investigated and was found to be stable. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated. Further the global stability is discussed using Liapunov's method[1]. Another work by P.J. Johnson in 2006 investigated when feasible trajectories exist and under what conditions the phenomena of permanence and competitive exclusion are exhibited in a discrete time system[2]. The analysis of the system was carried out using the Method of Critical Curves as well as traditional methods in the fields of nonlinear dynamics and population dynamics. Since its introduction, the Method of Critical Curves has proven to be an important tool in the analysis of the global dynamical properties of noninvertible maps[3]. By constructing a pair of ordered upper and lower solutions, the existence of nontrivial nonnegative periodic solutions was established for degenerate parabolic systems[4]. The initial boundary value problem of a two-species degenerate parabolic cooperative system was first considered. By using the method of a parabolic regularization and energy estimate, the existence of the weak solution of the problem was established. The main aim of this study is to understand what happens in both systems for all possible nonnegative initial conditions. We are also interested in the stability of these systems at the equilibrium points, the inhibition and coexistence of the species in a bounded domain.

The Models

Cooperative Species model: The cooperative species model is described by the system of differential equations where both species benefit by their interactions.

$$\frac{dx_1}{dt} = -\alpha_1 x_1 + \beta_1 x_1 x_2 \qquad (1)$$
$$\frac{dx_2}{dt} = -\alpha_2 x_2 + \beta_2 x_1 x_2$$
$$\mathbf{x_1 \geq 0, x_1 \geq 0}$$

Where $\boldsymbol{x_1}$ is the population of prey and $\boldsymbol{x_2}$ is the population of predator. The coefficient $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ are death rate of prey and predator without the interaction of each other (ie. the reliance of the two species upon each other). The greater the coefficient, the less valuable/critical one species is to the other; that is, $\boldsymbol{\alpha_1}$ closer to zero will have less reliance on the opposite species for continue growth. The coefficient $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ are constants of proportionality that measures the number of prey and predator benefited by their interaction. For all cooperative systems, if either population starts at 0, the system behaves as an exponential decay system. This is logical, since species realy on each other for growth in a cooperative system. We wish to understand both systems for all nonnegative initial conditions (starting populations); for this reason, only $\boldsymbol{x_1 \geq 0}$ and $\boldsymbol{x_2 \geq 0}$ are considered.

To determine equilibrium points for the system, we set $\frac{dx_1}{dt} =$

$\frac{dx_2}{dt} = 0.$

$$\frac{dx_1}{dt} = -\alpha_1 x_1 + \beta_1 x_1 x_2 = 0$$
$$\frac{dx_2}{dt} = -\alpha_2 x_2 + \beta_2 x_1 x_2 = 0$$

The equilibrium points are given as (0,0) and $(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1})$. Considering the initial conditions $x_1 = 0$, the solution of the system becomes $\frac{dx_2}{dt} = -\alpha_2 x_2$ and $\frac{dx_1}{dt} = 0$. This confirms the fact that the rate of change of population of predator decays exponentially. Thus $x_2(t) = Ke^{-\alpha_2 t}$

Similarly, when we consider the initial condition $x_2 = 0$, the solution of the system becomes $\frac{dx_1}{dt} = -\alpha_1 x_1$ and $\frac{dx_2}{dt} 0$. This also confirms the statement that the rate of change of population of preys decay exponentially. Thus $x_1(t) = Ce^{-\alpha_1 t}$. If there is no prey, there won't be predator and if there is no predator, there is no prey.

The Jacobian of matrix (1) is given as;

$$J(x_1, x_2) = \begin{bmatrix} -\alpha_1 + \beta_1 x_1 & \beta_1 x_1 \\ \beta_2 x_2 & -\alpha_2 + \beta_2 x_1 \end{bmatrix}$$

The eigenvalues(λ) are the roots of the characteristic polynomial $|A - \lambda I| = 0$. At the equilibrium point (0,0);

$$|A - \lambda I| = \begin{bmatrix} -\alpha_1 - \lambda & 0 \\ 0 & -\alpha_2 - \lambda \end{bmatrix} = 0$$

This yields the eigenvalues $\lambda_1 = -\alpha_1$and $\lambda_2 = -\alpha_2$. This is a stable node.

At $(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1})$;

$$J(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1}) = \begin{bmatrix} 0 & \frac{\beta_1 \alpha_2}{\beta_2} \\ \frac{\beta_2 \alpha_1}{\beta_1} & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} -\lambda & \frac{\beta_1 \alpha_2}{\beta_2} \\ \frac{\beta_2 \alpha_1}{\beta_1} & -\lambda \end{bmatrix} = 0$$

This also yields the eigenvalues $\lambda_{1,2} = \pm\sqrt{\alpha_1 \alpha_2}$ which is unstable node (That is a saddle).

Competitive Species Model: Competitive model is the one in which both species are harmed by each other. This is described by the system of differential equations;

$$\frac{dx_1}{dt} = \gamma_1 x_1 - \psi_1 x_1^2 - \phi_1 x_1 x_2 = \gamma_1 x_1 \left(1 - x_1 \left(\frac{\psi_1}{\gamma_1}\right)\right) - \phi_1 x_1 x_2 \tag{2}$$

$$\frac{dx_1}{dt} = \gamma_1 x_1 - \psi_1 x_1^2 - \phi_1 x_1 x_2 = \gamma_1 x_1 \left(1 - x_1 \left(\frac{\psi_1}{\gamma_1}\right)\right) - \phi_1 x_1 x_2$$

$\boldsymbol{x_1 \geq 0, x_1 \geq 0}$

In this model, both species have limited resources and they cannot grow without limit if there is no interaction between predator and prey. Therefore, carrying capacity for prey, which is $\frac{dx_1}{dt}$, is $\frac{\gamma_1}{\psi_1}$ and carrying capacity for predator, which is $\frac{dx_2}{dt}$, is $\frac{\gamma_2}{\psi_2}$. γ_1 and γ_2 are growth rates of prey (x_1) and predator (x_2) respectively. A number that measures prey, x_1 and predator, $x_2$harmed by their interactions is the coefficient of $x_1 x_2$, which are ϕ_1 and ϕ_2.

The equilibrium points are given as (0,0), $\left(\frac{\gamma_1}{\psi_1}, 0\right)$,$\left(0, \frac{\gamma_1}{\psi_1}\right)$ and $\left(\frac{\gamma_1\psi_2 - \gamma_2\phi_1}{\psi_1\psi_2 - \phi_1\phi_2}, \frac{\gamma_2\psi_2 - \gamma_1\phi_2}{\psi_1\psi_2 - \phi_1\phi_2}\right)$

At the initial condition, $x_1 = 0$, the solution of the system becomes $\frac{dx_1}{dt} = 0$ and $\frac{dx_2}{dt} = \gamma_2 x_2 - \psi_2 x_2^2$.

$\Rightarrow$

$$\int \frac{1}{\gamma_2 x_2} dx_2 + \int \frac{\psi_2}{\gamma_2(\gamma_2 - \psi_2 x_2} dx_2 = \int dt$$

$$\frac{1}{\gamma_2} \ln(\gamma_2 - \psi_2 x_2) - \frac{1}{\gamma_2} \ln\ x_2 = -t + C$$

$$x_2(t) = \frac{\gamma_2}{\psi_2 + Ce^{-t\gamma_2}}$$

The limit as time approaches infinity is given as;

$$x_2(t) = \lim_{t\to\infty} \frac{\gamma_2}{\psi_2 + Ce^{-t\gamma_2}} = \frac{\gamma_2}{\psi_2}$$

Therefore if $x_1 = 0$, the population of species $x_2 = 0$, the solution of the system becomes $\frac{dx_2}{dt} = 0$ and $\frac{dx_1}{dt} = \gamma_1 x_1 - \psi_1 x_1^2$. Solving the second differential equation also gives;

$$x_1(t) = \frac{\gamma_1}{\psi_1 + Ce^{-t\gamma_1}}$$

As t goes to infinity,

$$x_1(t) = \lim_{t\to\infty} \frac{\gamma_1}{\psi_1 + Ce^{-t\gamma_1}} = \frac{\gamma_1}{\psi_1}$$

This means if $x_2 = 0$, the population of species x_1 will tend to $\frac{\gamma_1}{\psi_1}$ as time goes to infinity.

The Jacobian matrix of (2) is given as;

$$J(x_1, x_2) = \begin{bmatrix} \gamma_1 - 2\psi_1 x_1 - \phi_1 x_2 & -\phi_1 x_1 \\ -\phi_2 x_2 & \gamma_2 - 2\psi_2 x_2 - \phi_2 x_1 \end{bmatrix}$$

At equilibrium point (0,0);

$$|A - \lambda I| = \begin{bmatrix} -\gamma_1 - \lambda & 0 \\ 0 & -\gamma_2 - \lambda \end{bmatrix} = 0$$

The resulting eigen values are $\lambda_1 = -\gamma_1$ and $\lambda_2 = -\gamma_2$ which is again, a stable node.

At the equilibrium point $\left(\frac{\gamma_1}{\psi_1}, 0\right)$;

$$J = \begin{bmatrix} -\gamma_1 & \frac{-\phi_1\gamma_1}{\psi_1} \\ 0 & \gamma_2 - \frac{\phi_2\gamma_1}{\psi_1} \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\gamma_1 - \lambda & \frac{\phi_1\gamma_1}{\psi_1} \\ 0 & \gamma_2 - \frac{\phi_2\gamma_1}{\psi_1} - \lambda \end{vmatrix} = 0$$

The roots of this quadratic form are $\lambda_1 = -\gamma_1$and $\lambda_2 = \gamma_2 - \frac{\phi_2\gamma_1}{\psi_1}$. This equilibrium point is locally asymptotically stable if $\lambda_2 < 0$, that is if $\frac{\gamma_1}{\psi_1} > \frac{\gamma_2}{\phi_2}$

At the equilibrium point $\left(0, \frac{\gamma_2}{\psi_2}\right)$;

$$J = \begin{bmatrix} \gamma_1 - \frac{\phi_1\gamma_2}{\psi_2} & 0 \\ -\frac{\phi_2\gamma_2}{\psi_2} & -\gamma_2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} \gamma_1 - \frac{\phi_1\gamma_2}{\psi_2} - \lambda & 0 \\ -\frac{\phi_2\gamma_2}{\psi_2} & -\gamma_2 - \lambda \end{vmatrix} = 0$$

This also yields the eigenvalues $\lambda_2 = -\gamma_2$and $\lambda_1 = -\gamma_1 - \phi_1 \frac{\gamma_2}{\psi_2}$. This equilibrium point is locally asymptotically stable if $\lambda_1 < 0$, that is if $\frac{\gamma_1}{\phi_1} < \frac{\gamma_2}{\psi_1}$ [5].

At the equilibrium point $\left(\frac{\gamma_1\psi_2-\gamma_2\phi_1}{\psi_1\psi_2-\phi_1\phi_2}, \frac{\gamma_2\psi_2-\gamma_1\phi_2}{\psi_1\psi_2-\phi_1\phi_2}\right)$, the Jacobian matrix is given by;

$$J = \begin{bmatrix} \gamma_1 - 2\psi_1 \frac{\gamma_1\psi_2-\gamma_2\phi_1}{\psi_1\psi_2 - \phi_1\phi_2} - \phi_1 \frac{\gamma_2\psi_1-\gamma_1\phi_2}{\psi_1\psi_2 - \phi_1\phi_2} & \\ \phi_2 \frac{\gamma_2\psi_1-\gamma_1\phi_2}{\psi_1\psi_2 - \phi_1\phi_2} & \end{bmatrix}$$

$$J = \begin{bmatrix} & -\phi_1 \frac{\gamma_2\psi_1-\gamma_1\phi_2}{\psi_1\psi_2 - \phi_1\phi_2} \\ & \gamma_2 - 2\psi_2 \frac{\gamma_2\psi_1-\gamma_1\phi_2}{\psi_1\psi_2 - \phi_1\phi_2} - \phi_1 \frac{\gamma_1\psi_2-\gamma_2\phi_1}{\psi_1\psi_2 - \phi_1\phi_2} \end{bmatrix}$$

The trace of this matrix is given as

$$trace(J) = \frac{\gamma_1\psi_1\psi_2 - \psi_2\gamma_2\phi_1 + \psi_2\psi_1\psi_2 - \psi_2\gamma_1\phi_2}{\psi_1\psi_2 - \phi_1\phi_2}$$

With determinant

$$\det(J) = \frac{(\gamma_1\psi_2 - \phi_1\gamma_2)(\psi_1\gamma_2 - \phi_2\gamma_1)}{\psi_1\psi_2 - \phi_1\phi_2}$$

This critical point exists provided $\psi_1\psi_2 \neq \phi_1\phi_2$. The stability of this point therefore depends on whether $\psi_1\psi_2 > \phi_1\phi_2$ or $\psi_1\psi_2 < \phi_1\phi_2$[6]. The expression $\phi_1\phi_2$ is a measure of inhibition while $\psi_1\psi_2$ is a measure of competiton among the two species.

When inhibition is greater than competition, $\psi_1\psi_2 > \phi_1\phi_2$, among the two competing species, and if $\frac{\gamma_2}{\phi_2} > \frac{\gamma_1}{\phi_1}$and $\frac{\gamma_2}{\phi_2} > \frac{\gamma_1}{\psi_1}$ then $\psi_1(\gamma_2\phi_1 - \gamma_1\phi_1) > 0$ and $\psi_2(\gamma_2\phi_1 - \gamma_1\phi_2) > 0$. Hence, the $trace(J) < 0$, and the $\det J > 0$**.** Therefore, the critical point is locally asymptotically stable, and the two species can coexist at this point.

Similarly, when inhibition is less than competition, $\psi_1\psi_2 < \phi_1\phi_2$, among the two competing species, and if $\frac{\gamma_2}{\phi_2} > \frac{\gamma_1}{\phi_1}$ and. $\frac{\gamma_2}{\phi_2} > \frac{\gamma_1}{\psi_1}$ then $\psi_1(\gamma_2\phi_1 - \gamma_1\phi_1) < 0$ and $\psi_2(\gamma_2\phi_1 - \gamma_1\phi_2) < 0$. Hence, the $trace(J) > 0$. Therefore, the critical point is unstable, hence the two species cannot coexist

Analysis and Discussion

Cooperative Species model: Phase diagrams were drawn to illustrate the flow of trajectories of the models. From the figure 1, the vector field points to the equilibrium point (0,0) along the curves in both horizontal and vertical vector fields. If we consider the long-term behavior of the population, there will be different population evolution scenario depended on different initial conditions. In the figure, solution curve corresponding to the above initial condition, we could conclude that the solution approaches the equilibrium poin t (0,0). Both population of predator and prey will die out as t goes to infinity.

Even population of x_1 species starts out larger amount of population, they become vanish without x_2 species because species x_2 is always decreasing in this case. Population of species x_2 increases for a short time and after that both population of species x_2 and x_1 decline to equilibrium point (0,0).

Population of x_1 decreases for a short period while population x_2 increases. After that population x_1 gets enough supply since population of species x_2 increases the whole time and then population of species x_1 grows and so does population of species x_2. Both population species $x_1$and x_2 tend to go to equilibrium point $\left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1}\right)$, however, they are cooperative species and they are benefited by their interaction of each other and they start to grow without bound.

Competitive Species Model: By analyzing figure 2, it is obvious that population of species x_2 vanished while the population of species x_1 tends to increase and stabilize at the equilibrium point $\left(\frac{\gamma_1}{\psi_1}, 0\right)$. Population of x_2 species tries to increase while x_1 species is increasing and they can't make it.

Both $x_1$and x_2 population is decreasing at first and then x_1 population recovers and increases to stabilize and x_2 species will die out. In the graphs, it is easy to realize that x_1 population becomes extinct and $x_2$population increase to stabilize $x_2 = \frac{\gamma_2}{\psi_2}$. In this case population $x_2$species is increasing and tends to equilibrium point $\left(0, \frac{\gamma_2}{\psi_2}\right)$ and x_1 species try to increase for short time and after that it is decreasing to become extinct. In the figure, x_1 and x_2 population starts out almost the same value and however, x_1 population dies out and $x_2$population is approaching to equilibrium point $\left(0, \frac{\gamma_2}{\psi_2}\right)$.

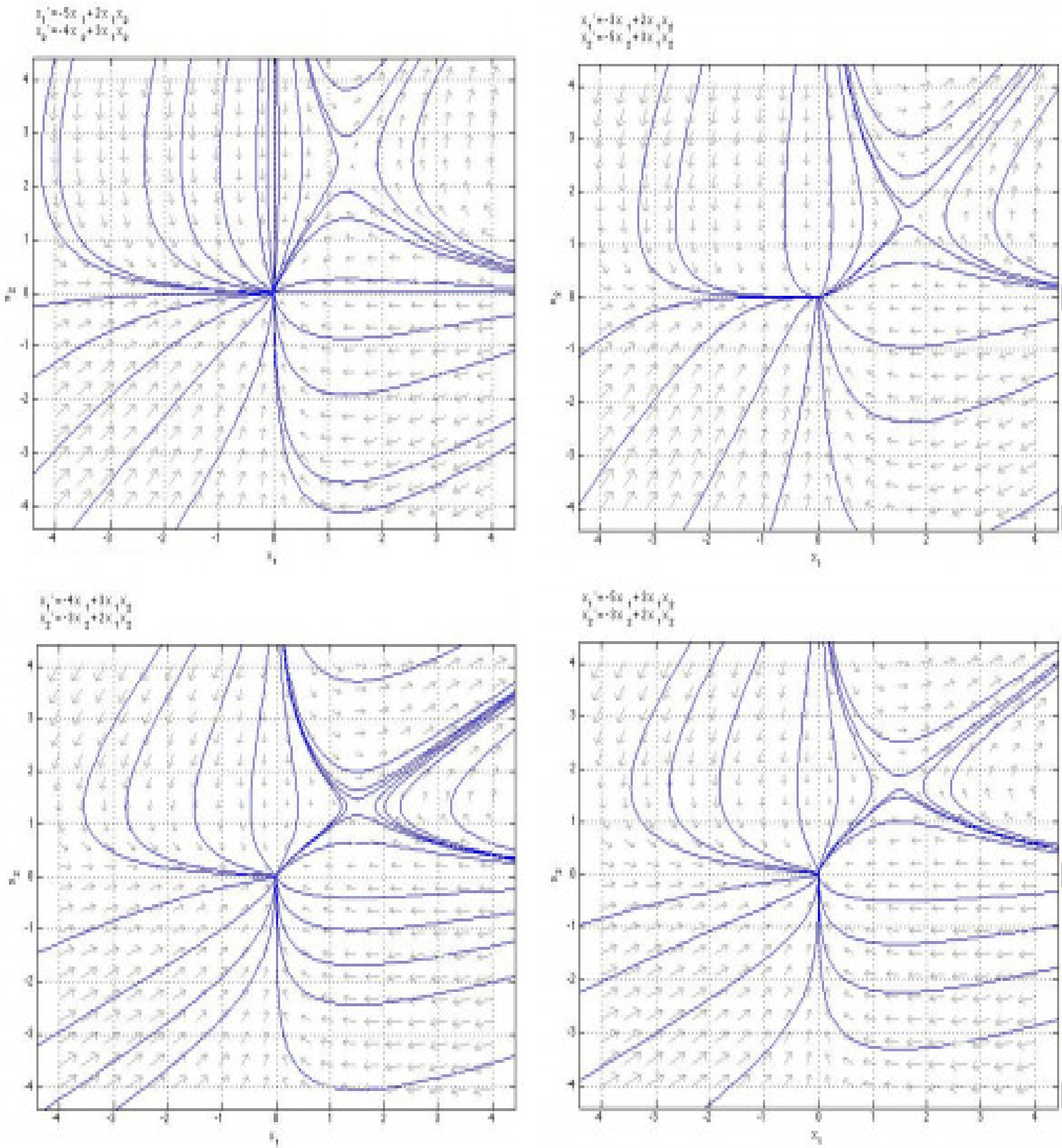

Figure-1
Phase Portrait for Cooperative Species Model

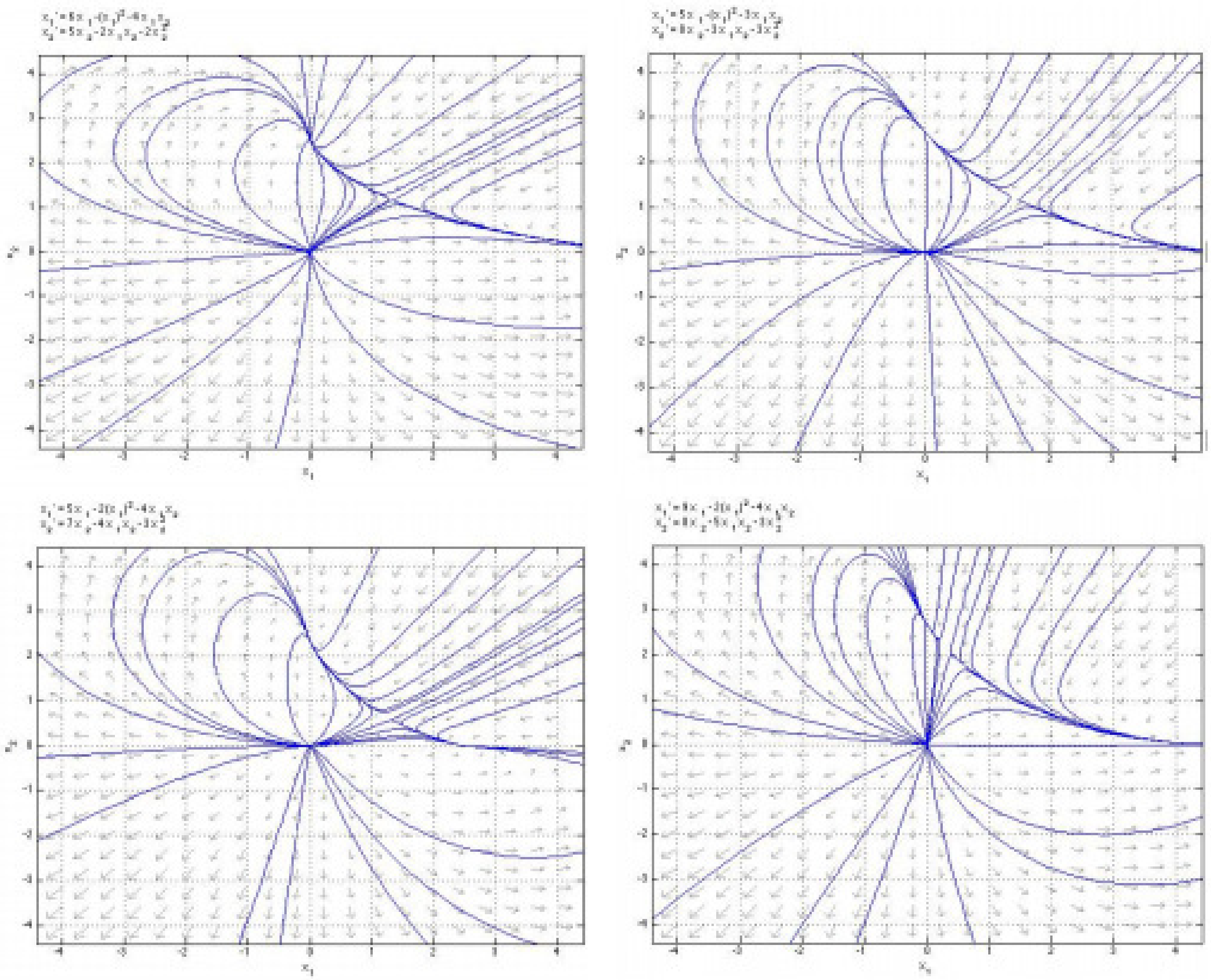

Figure-2
Phase Portrait for Competitive Species Models

Conclusion

In this study, a stability analysis on a generalised mathematical models for cooperative and competitive species was performed. The equilibrium for each system was determined and the behaviour of solutions whose initial conditions satisfy either species, $x_1 = 0$, or species, $x_2 = 0$, were analysed. Curves in the phase plane along which the vector field is either horizontal or vertical were presented and explained. For each of the systems, we described all possible population scenarios using the phase portraits as a comparison basis. The cooperative system was found to have two equilibrium points of which one is stable and the second one being a saddle point.

Four equilibrium points exist for the competitive system which are stable for one point and conditional locally asymptotically stable for the rest. Based on the inhibition and the competition factors between the two competing species, one of these equilibrium points is locally asymptotically stable or unstable. When inhibition is greater than the competition among the two competing species, the system becomes locally asymptotically stable, and the two species can coexist at this point. Otherwise, it is unstable and the two species cannot coexist.

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