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α-Sasakian Manifolds Admitting Ricci Soliton

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Abstract

In this paper we study η-Einstein α-Sasakian manifolds admitting Ricci soliton.

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Introduction

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M, g), g is called a Ricci soliton studied by Hamilton[1] if

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad (1)$$

where $\pounds$ is the Lie derivative, S is the Ricci tensor, λ is a constant and V is a potential vector field on M. Metrics satisfying (1) are interesting and useful in physics and are often referred as quasi-Einstein. Compact Ricci soliton are special case of the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2S_{ij}$ with fixed point. There are many authors Perelman[2] which study compact Ricci soliton and obtain many good results.

If λ is negative the Ricci soliton is said to be shrinking, if λ is zero the Ricci soliton is said to be steady and if λ is positive the Ricci soliton is said to be expanding. g is said to be a gradient Ricci soliton if the vector field V is the gradient of a potential function $-f$ and the equation (1) has written of the form $\nabla\nabla f = S + \lambda g$. In dimension 2 and 3, a Ricci soliton on a compact manifold has constant curvature. For detail we refer Chow and Knopf[3] and Derdzinski[4].

If Ricci tensor of α-Sasakian manifolds is written like (2) then it is η-Einstein manifold which is given as

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \quad (2)$$

where a and b are constant for $n > 1$, Zhang[6] studied compact Sasakian manifold with constant curvature and quasi-positive holomorphic bisectional transverse curvature. Sharma and Ghosh[7] show that, if a 3-dimensional Sasakian metric is a non trivial Ricci soliton, then it is homothetic to the standard Sasakian structure on Heisenberg group nil[3]. A K-contact manifold is Sasakian manifold in dimension 3 which is not true in higher dimension.

This paper organised as follow:

Section 2, is devoted to preliminaries definition of α-Sasakian manifold and some properties of α-Sasakian manifold. In section 3, we have a theorem and an example of a Sasakian-space form (generalized) $M(f_1, f_2, f_3)$ with $f_1 = (c + 3\alpha^2)/4$ and $f_2 = f_3 = (c - \alpha^2)/4$. Also, it is η-Einstein, and fallows all the conclusion of the theorem and M is $R^{(2n+1)}(\alpha^2 - 4)$ recognizable with the $(2n + 1)$-dimensional Heisenberg group.

α-Sasakian Manifolds

A contact manifold is a $(2n + 1)$-dimensional C^∞ manifold M equipped with a global form η, called a contact form of M such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. In particular, $\eta \wedge (d\eta)^n \neq 0$ is a volume element of M so that a contact manifold is orientable. A contact manifold associated with the Riemannian metric g is called contact metric manifold if it satisfy the following relation (3)

$$d\eta(X, Y) = g(X, \phi Y), \eta(X) = g(X, \xi), \ \phi^2 = -I + \eta\otimes\xi, \quad (3)$$

Where ϕ is a (1, 1)-tensor field and ξ is a unique vector field such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. We denote the symbols ∇, R and Q by Levi-Civita connection, curvature tensor and Ricci operator of g respectively. We define a (1, 1) type tensor field h by $h = \frac{1}{2}\pounds_\xi\phi$ and we know that h and $h\phi$ are trace free and $h\phi = -\phi h$. We define an operator l by $lX = R(X, \xi)\xi$ for all X. Then obviously $l\xi = 0$ and l is a self-adjoint operator. contact metric manifolds has following properties,

$$\nabla_X \xi = -\phi X - \phi h X, \quad (4)$$
$$l - \phi l \phi = -2(h^2 + \phi^2), \quad (5)$$
$$\nabla_\xi h = \phi - \phi l - \phi h^2, \quad (6)$$
$$Tr.\, l = S(\xi, \xi). \quad (7)$$

An almost contact manifold $M(\phi, \eta, \xi, g)$ is trans-Sasakian manifold if there exist two function α and β on M such that

$$(\nabla_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$

for any vector X, Y on M. If $\beta = 0$ then M is α-Sasakian manifold. Sasakian manifolds is a case of α-Sasakian manifold with $\alpha = 1$. If $\alpha = 0$ then M is called β-Kenmotsu manifold. Kenmotsu manifolds are case of β-Kenmotsu with $\beta = 1$. If both α and β vanish, then M is a cosymplectic manifold. Here we consider α-Sasakian manifold and following holds in α-Sasakian manifold,

$$\nabla_X\xi = -\alpha\phi X, \tag{8}$$
$$R(X,\xi)\xi = \alpha\{X - \eta(X)\xi\}, \tag{9}$$
$$Q\xi = 2n\xi\alpha, \tag{10}$$
$$(\nabla_X\phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\}, \tag{11}$$

Theorems and Example

Theorem: If the η-Einstein (non-Einstein) α-Sasakian manifold $M(\phi,\eta,\xi,g)$ has Ricci soliton (non-trivial) with potential vector V, then
(i). Jacobi along the geodesics is V which is determine by ξ. (ii). V is infinitesimal contact transformation which depend on value of α. (iii). The Ricci soliton is expanding.

Proof: We take M is η-Einstein then from (2) we can get the value of r which is given by,

$$r = (2n+1)a + b, \tag{12}$$

now using (2) in (1) we get,

$$(\pounds_V g)(Y,Z) = -2(\lambda + a)g(Y,Z) - 2b\eta(Y)\eta(Z), \tag{13}$$

Differentiating (13) with respect to vector field X and then applying (8) we have,

$$(\pounds_V\nabla_X g)(Y,Z) = 2b\alpha[g(Y,\phi X)\eta(Z) + g(Z,\phi X)\eta(Y)], \tag{14}$$

now we taking use of Yano[8] (1970) formula which is given as,

$$(\pounds_V\nabla_X g - \nabla_X\pounds_V g - \nabla_{[V,X]}g)(Y,Z) = -g((\pounds_V\nabla)(X,Y),Z) - g((\pounds_V\nabla)(X,Z),Y),$$

We obtain,

$$(\nabla_X\pounds_V g)(Y,Z) = g((\pounds_V\nabla)(X,Y),Z) + g((\pounds_V\nabla)(X,Z),Y), \tag{15}$$

now, use of (14) in (15) and a straightforward combinatorial computational shows

$$(\pounds_V\nabla)(Y,Z) = 2b\alpha[\eta(Z)\phi Y + \eta(Y)\phi Z], \tag{16}$$

substituting $Y = Z = \xi$ in (15) we have, $(\pounds_V\nabla)(\xi,\xi) = 0$. In the formula of Duggal and Sharma[9] (1999), we using above and ξ is geodesic [we can see from (8)] we have,

$$(\pounds_V\nabla)(X,Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y}V + R(V,X)Y,$$

gives that $\nabla_\xi\nabla_\xi V + R(V,\xi)\xi = 0$, which implies that Jacobi along the geodesics is V which is determine by ξ, which is (i)

Next, differentiating (15) with respect to vector field X and then applying (8) we have,

$$(\nabla_X\pounds_V\nabla)(Y,Z) = 2b\alpha\{-\alpha g(Z,\phi X)\phi Y - \alpha g(Y,\phi X)\phi Z + \eta(Z)(\nabla_X\phi)Y + \eta(Y)(\nabla_X\phi)Z\}, \tag{17}$$

making use of the (16) and the identity,

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z),$$

one obtains

$$(\pounds_V R)(X,Y)Z = 2b\alpha[-\alpha g(Z,\phi X)\phi Y + \alpha g(Z,\phi Y)\phi X + 2\alpha g(X,\phi Y)\phi Z + \eta(Z)\{(\nabla_X\phi)Y - (\nabla_Y\phi)X\} + \eta(Y)(\nabla_X\phi)Z - \eta(X)(\nabla_Y\phi)Z], \tag{18}$$

setting $Y = Z = \xi$ in (18) shows that,

$$(\pounds_V R)(X,\xi)\xi = 4\alpha b[\eta(X)\xi - X], \tag{19}$$

next Lie differentiation (9) along V and using (19) and (13) we get,

$$4\alpha b[\eta(X)\xi - X] + R(X,\pounds_V\xi)\xi + R(X,\xi)\pounds_V\xi = \alpha\{-\eta(X)\pounds_V\xi - g(X,\pounds_V\xi)\xi + 2(\lambda + a + b)\eta(X)\xi\}, \tag{20}$$

Contracting (20) over X and $g(\pounds_V\xi,\xi) = (\lambda + a + b)$ (follows from (13) by taking $Y = Z = \xi$) gives

$$a - b + \lambda = 0, \tag{21}$$

now we use integrability condition of the Ricci soliton we get,

$$\pounds_V r = -\Delta r + 2\lambda r + 2|S|^2, \tag{22}$$

Where $\Delta r = -\mathrm{div.}Dr$. Comparing the value of $|S|^2$ from (2) and using (10) and (12) we find that, $b(a+2) = 0$. Since $b \neq 0$, because if $b = 0$ then M is Einstein which is a contradiction hence we have,
$a = -2$ and $b = 2(n+1)$.

Thus, it follows that $\lambda = 2(n+2) > 0$, which show that Ricci soliton is expanding, which prove part (ii) of the theorem.

Contracting (18) along X and using the formula $(\mathrm{div}\phi)X = -2n\eta(X)$ for a contact metric one gets

$$(\pounds_V S)(Y,Z) = 4\alpha b[g(Y,Z) - (2n+1)\eta(Y)\eta(Z)], \tag{23}$$

next, in (2) we take the Lie- derivative of $S(X,Y)$ along V and then using (13) we get,

$$(\pounds_V S)(Y,Z) = -2(a^2 + a\lambda)g(Y,Z) + b[(\pounds_V\eta)(Y)\eta(Z) + \eta(Y)(\pounds_V\eta)Z] - 2ab\eta(Y)\eta(Z), \tag{24}$$

comparing above two equations and put $Z = \xi$, and substituting the value of a, b and λ obtained above, we get $\pounds_V\eta = -4(n+\alpha)$, V is infinitesimal contact transformation which depends on the value of α, which is the part (iii) of the theorem. Also by the straight forward calculation, we find that $\pounds_V\xi = 4(n+\alpha)\xi$. Thus proof of the theorem is complete.

Example

A $M(f_1, f_2, f_3)$ generalized Sasakian-space-form which is α-Sasakian manifold with $f_1 = (c + 3\alpha^2)/4$ and $f_2 = f_3 = (c - \alpha^2)/4$. Also, it is η-Einstein hence it follow the theorem. The value of a and b for generalized Sasakian-space-form are $a = \frac{n(c+3\alpha^2)+(c-\alpha^2)}{2}$ and $b = \frac{-(c-\alpha^2)(n+1)}{2}$. Now from these values, comparing the values of a and b which get from the theorem, we get $c = \alpha^2 - 4$. Thus $M(f_1, f_2, f_3)$ is $R^{2n+1}(\alpha^2 - 4)$ identifiable with the $(2n+1)$-dimensional Heisenberg group. This prove the corollary. Hence M is $R^{(2n+1)}(\alpha^2 - 4)$ recognizable with the $(2n+1)$-dimensional Heisenberg group.

Conclusion

In this paper we study α-Sasakian manifold whose metric manifolds whose metric manifolds whose metric as Ricci soliton and we can see that when it is non-trivial Ricci soliton with potential vector V then Ricci soliton is expanding, V is Jacobi along geodesics determine by ξ and V is infinitesimal contact transformation.

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