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The Solution Elements of Pure Subgroup is Random

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Abstract

In this paper, we are study the new definition of the pure subgroups of finite abelian group; which is called pure solutions element of pure subgroups, and we get a new results of random element and we are study the diameter of "Cayley" graphs of the abelian groups. Moreover, we get new results of symmetric groups..

Keywords: Solution, elements, subgroup, random.

Introduction

A subgroup H of finite abelian group, is pure if$\forall n \in Z, \forall x \in H\, if\ \ n|x\ in\, G\, then\ \ n|x\ in\, H$. Which means that $\exists$ an element $h \in H\ \ni nh = x \in H$. The element h is said to be pure solution element in H. A random solution is a random element of H.
A random pair is random solution of the $H \times H = \{(h_1, h_2), h_1, h_2 \in H,$ and are solutions elements of the equation $nh = x \in H\}$.

Example (1): Let $G = Z_3 \otimes Z_4$ and take
$H = \{(0,0), (0,2), (1,0), (1,2), (2,0), (2,2)\}$
Test the element $(0,2)$ in H
Take $n = 2$, clearly $2|(0,2)$ in G
$(2(0,1) = (0,2) \in H)$but $2|(0,\ 2)$ in H. There is no element $\in H(say\ x) \ni 2(0,x) = (0,2)$ so the element $(0,1) \in G$ is not pure solutionin H.

Example (2): Take $G = Z_4 \otimes Z_6$ and
$H = \{(0,0), (0,3), (2,0), (2,3)\}$
Take $n = 3$, clearly that $3|(0,3)$ in G
$3(0,1) = (0,3)$ and $3|(0,3)$ in $H, \big(3(0,3) = (0,3)\big)$ we can show that $\forall n \in Z^+, n$ is odd number then $n(0,3) = (0,3) \Rightarrow n|(0,3)$ in H, but $\forall n \in Z^+, n$ is even number $n|(0,3)$ in G so $n|(0,3)$ in H therefore $\forall n \in Z^+, n|(0,3)$ in H.
the element $(0,3)$ is saidto be pure solution in H.
The study of the diameter of cayley graphs of the groups[1-3]. It can also be viewed as a contribution by Erdo and Turn[4]. Here we assume G is symmetric group. We also observe and gave a new results of pure random solutions.

Theorem A. Let H be a pure subgroup of G has (h_0) fixed points, then the subgroup generated by H and a random pure has fixed solution is $\geq {}^{h_0}/_{2n}$, with $t, k\ \geq 2$ and $1 - \frac{t}{\binom{n}{k}} - f(n,k,t)$ (1)

Where $f(n,k,t)\ = \begin{cases} 0 & if\ k > n/4 \\ \frac{\binom{t}{2}\left(1+0\left(\frac{1}{n}\right)\right)}{\binom{n}{2k}} & if\ k \leq n/4 \end{cases}$ (2)

Proof. Let $|G| \leq n$ and take $k \leq n/2$,
(here$0(G) \geq 4$) and $t \leq n/k$.
Let $\# = \pi(H) = (x_1, x_2, \ldots, x_k)$ be the set of all pure solutions of H.And suppose that q is probability of $H$and P is probability of B, where B is another pure subgroup of G.

So $(B) = \frac{1}{\binom{n}{|B|}}$, using the union bounded

$q(H) \leq \sum_{r=1}^{t-1} \sum_{B \leq T_r} pB\ ,\ |B| \leq \frac{n}{2}\quad so\quad |T_r| \leq \binom{t}{r}$ (3)

Moreover, for $B \leq T$, we have $rk \leq \frac{n}{2}$

Therefore;

$q(H) \leq \sum_{r=1}^{n/2k} \frac{\binom{t}{r}}{\binom{n}{rk}} \leq \frac{t}{\binom{n}{k}} + f(n,k,t)$ (4)

The last inequality is vacuously true if $k > {}^{n}/_{6}$, the case $k \leq \frac{n}{6}$ is the content of the next theorem.

Theorem B. Let $2 \leq k \leq {}^{n}/_{6}$ and $tk \leq n$ with H is pure in G.

$\sum_{r=2}^{n/2k} \frac{\binom{t}{r}}{\binom{n}{rk}} = o\left(\frac{\binom{t}{2}}{n\binom{n}{2k}}\right)$ (5)

Proof. Since G is finite abelian group and H pure in G then H has finite random solutions so the set of solutions of $H \leq o(G)$.
Let $a_r = \binom{t}{r}$ and $b_r = \binom{n}{rk}$ and $f(n,k,t) = \sum_{r=2}^{[n/2k]} {}^{(b_2 a_r)}\big/_{a_2 b_r}$

Claim $nf(n,k,t)$is bounded.
We observe that

$\left(\binom{t}{r}\right)^k \leq \binom{tk}{rk} \leq \binom{n}{rk}$ (6)

Further we observe that for $n \geq 64$ and $rk \leq {}^{n}/_{2}$, we have

$$\binom{n}{rk} > \left(\binom{n}{2k}\right)^4 \quad (7)$$

$$\text{Indeed } \binom{n}{64k} > \left(\frac{n}{64k}\right)^{64k} > \left(\frac{en}{2k}\right)^{8k} > \left(\binom{n}{2k}\right)^4 \quad (8)$$

Combining inequalities (6) and (7), we obtain for $r \geq 64$ that

$$\frac{b_2 a_r}{b_r} < \frac{1}{b^2} \leq \frac{1}{\binom{n}{4}} < \frac{1}{n^2} \quad (9)$$

It follows that

$$f_1(n,k,t) = \sum_{r=64}^{[n/2k]} \frac{b_2 a_r}{a_2 b_r} < \frac{1}{n} \quad (10)$$

It remains to bounded the sum

$$f_2(n,k,t) = \sum_{r=2}^{m} \frac{b_2 a_r}{a_2 b_r} \quad (11)$$

Where $m = \min\left\{63, \left[{}^{n}/_{2k}\right]\right\}$

Obviously,

$$f_2(n,k,t) \leq \sum_{r=2}^{m} \frac{b_2 a_m}{b_3} < \frac{n^{64} b_2}{b_3} \quad (12)$$

So,

$$\frac{b_2}{b_3} < \left(\frac{3k}{n-2k}\right)^k \quad (13)$$

Since $k \leq {}^{n}/_{6}$, the right hand side is less than $\left(\frac{3}{4}\right)^k$, so we obtain the estimate $f_2(n,k,t) < {}^{n^{64}}\big/_{\left(\frac{3}{4}\right)^k} \leq \frac{1}{n}$

If $k \geq {}^{65 logn}\big/_{\log\left(\frac{4}{3}\right)}$.

Assume now that $k \geq {}^{65 logn}\big/_{\log\left(\frac{4}{3}\right)}$.

It follows that for large enough n we have ${}^{3k}/_{(n-2k)} < {}^{1}/_{\sqrt{n}}$ and so

$$f_2(n,k,t) < {}^{n^{64} b_2}\big/_{b_3} < n^{64} n^{-n} \leq \frac{1}{n}$$

Assuming $k \geq 130$

Now, let us assume $k \leq 129$.

Then,

$$\frac{b_2 a_r}{a_2 b_r} = o\left(\frac{t^{r-2}}{n^{k(r-2)}}\right) = o\left(n^{-(k-1)(r-2)}\right) = o\left(\frac{1}{n}\right) \ldots . (14)$$

Proving $f_2(n,k,t) = o\left(\frac{1}{n}\right)$

Theorem C. Let H be a pure in a permutation group G with $f \leq {}^{n}/_{2}$ fixed points. Let x_0 be a random solution in G. Then the probability that H and x_0 do not generate transitive group less that $(f+1)\left(\frac{1}{n+o\left(\frac{1}{n^2}\right)}\right)$.

Proof. Let $A = fix(H)$, so $|A| = f$. The probability that $a^{\emptyset} \neq B \subseteq A$ is invariant under x_0, as be for $pB = \frac{1}{\binom{n}{|B|}}$. Let $i(A)$ denote the probability that such an invariant nonempty subset exists. By the union bound

$$i(A) = \sum_{B \subseteq A} pB = \sum_{r=1}^{f} \frac{\binom{f}{n}}{\binom{n}{r}} = \frac{f}{n} + o\left(\left(\frac{f}{n}\right)^2\right) \quad (15)$$

Let now H_0 denote the subgroup generated by $H$and x_0, and let $R = {}^{\frown}/_{A}$ (the domain where the pure H actually acts).

Let $x_0 \in R$ be the projection of $x_0 \longrightarrow R$

By observation 12, tow solutions element $x, y \in R$ belong to the same orbit under H ⇌they belong to the same orbit of the group generated by the pure of solutions random H, and $x_0 \in R$.

Therefore, the probability that not all solutions element of R are in the same orbit under H_0 is

$$\leq (\frac{1}{(n-f)} + (0(\frac{1}{(n-f)^2} = \frac{1}{n} + 0\left({}^{(f+1)}/_{n^2}\right)$$

Finally, the probability that H_0 is not transitive is at most the sum of this quantity and $i(A)$ which in turn is

${}^{(f+1)}/_{n} + 0\left({}^{(f+1)}/_{n^2}\right)$

Conclusion

In this work we get a new results of random element and we study the diameter of "Cayley" graphs of the abelian groups. Moreover, we get new results of symmetric groups and our main results are theorem A, B and C.

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