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Integral Formulae's Involving Two H̄-function and Multivariable's General Class of Polynomials

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Abstract

The aim of the present paper is to derive a new Integral formulae's for the $\overline{H}$ -function due to Inayat-Hussain whose based upon some integral formulae due to Qureshi et.al. The results are obtained in a compact form containing the multivariable Polynomials.

Keywords: H̄ function, general class of polynomials, generalized Wright hypergeometric function.
GJSFR-F Classification: (MSC 2000) 33C45, 33C60.

Introduction

In 1987, Inayat-Hussain [1,2] introduced generalization form of Fox's H-function, which is popularly known as $\overline{H}$ -function. Now $\overline{H}$ -function stands on fairly firm footing through the research contributions of various Auhors[1-6,9].

$\overline{H}$ -function is defined and represented in the following manner[5]:

$$\overline{H}^{m,n}_{p,q}\left[z\right]=\overline{H}^{m,n}_{p,q}\left(z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_L z^{\xi}\,\overline{\phi}(\xi)d\xi \tag{1}$$

Where ($z\neq 0$) and

$$\overline{\phi}(\xi)=\frac{[\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)][\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)]^{A_j}}{[\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)]^{B_j}\,[\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)]} \tag{2}$$

It may be noted that $\overline{\phi}(\xi)$ the contains fractional powers of some of the gamma function and m,n,p,q are integers such that $1\leq m\leq q$, $1\leq n\leq p$, $(\alpha_j)_{1,p}$, $(\beta_j)_{1,q}$ are positive real numbers and $(A_j)_{1,n}$, $(B_j)_{m+1,q}$ may take non-integer values, which we assume to be positive for standardization purpose. $(\alpha_j)_{1,p}$ and $(\beta_j)_{1,q}$ are complex numbers.

The nature of contour L, sufficient conditions of convergence of defining integral (1) and other details about the $\overline{H}$-function can be seen in the papers[4-5]. The behavior of the $\overline{H}$-function for small values of |z| follows easily from a result given by Rathie[3]:

$$\overline{H}^{m,n}_{p,q}\left[z\right]=o(\left|z\right|^{\alpha}),\ \text{Where } \alpha=\min_{1\leq j\geq m}\operatorname{Re}\left(\frac{b_j}{\beta_j}\right),\ \left|Z\right|\to 0 \tag{3}$$

$$\Omega=\sum_{j=1}^{m}\left|B_j\right|+\sum_{j=m+1}^{q}\left|b_jB_j\right|-\sum_{j=1}^{n}\left|a_jA_j\right|-\sum_{j=1}^{m}\left|A_j\right|>0,\ 0<\left|Z\right|>\infty \tag{4}$$

The multivariable's general Class of polynomials defined and represented as follows[10]:

$$S^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}\left[x_1,\ldots,x_r\right]=\sum_{k_1}^{n_1/m_1}\cdots\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}\left\{\frac{(-n)_{m_ik_i}}{\lfloor k_i}A(k_1,\ldots,k_r)\,x_i^{k_i}\right\} \tag{5}$$

Where: $n_i, m_i=1,\ldots; m_i\neq 0, \forall i\in 1,2,\ldots,r$; the coefficients $A(k_1,k_2,\ldots,k_r)$, $(k_i\geq 0)$ are arbitrary constant, real or complex. The general class of polynomials[11] is capable of reducing to a number of familiar multivariable polynomials by suitable specializing the arbitrary coefficients $A(k_1,k_2,\ldots,k_r)$, $(k_i\geq 0)$

The general class of multivariable polynomials is defined by Srivastava and Garg[1]

$$S^{h_1,\ldots,h_r}_{L}\left[X_1,\ldots,X_r\right]=\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\leq L}(-L)_{h_1k_1+\ldots+h_rk_r}A(L,k_1,\ldots,k_r)\,\frac{X_1^{k_1}}{\lfloor k_1}\cdots\cdots\frac{X_r^{k_r}}{\lfloor k_r} \tag{6}$$

Where $h_1,h_2,\ldots,h_r$ are arbitrary positive integers and $A(L;k_1,k_2,\ldots,k_r)$, $(L;h_i\geq N; i=1,2,\ldots,r)$ coefficients are arbitrary constant, real or complex.

The following formulas[12] will be required in our investigation.

$$\int_0^\infty \left[(ax+\frac{b}{x})^2+c\right]^{-p-1}dx = \frac{\sqrt{\pi}}{2a(4ab+c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)}A$$

$$\left(a>0\,;\,b\geq 0\,;\,c+4ab>0\,;\,\mathrm{Re}(\rho)+\frac{1}{2}>0\right) \qquad (7)$$

$$\int_0^\infty \frac{1}{x^2}\left[(ax+\frac{b}{x})^2+c\right]^{-p-1}dx = \frac{\sqrt{\pi}}{2b(4ab+c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)}$$

$$\left(a\geq 0\,;\,b>0\,;\,c+4ab>0\,;\,\mathrm{Re}(\rho)+\frac{1}{2}>0\right) \qquad (8)$$

$$\int_0^\infty (a+\frac{b}{x^2})\left[(ax+\frac{b}{x})^2+c\right]^{-p-1}dx = \frac{\sqrt{\pi}}{(4ab+c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)}$$

$$\left(a>0\,;\,b>0\,;\,c+4ab>0\,;\,\mathrm{Re}(\rho)+\frac{1}{2}>0\right) \qquad (9)$$

Main Integral Formulae's

Let X stands for $[\,(ax+\frac{b}{x})^2+c\,]$

First Integral Formula:

$$\int_0^\infty X^{-\eta-1} S^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}\left[c_1X^{\delta_1},\ldots,c_rX^{\delta_r}\right] \bar{H}^{M,N}_{P,Q}\left[X^{\sigma}\left|\begin{matrix}(a'_j,\alpha'_j;A'_j)_{1,N},(a'_j,\alpha'_j)_{N+1,P}\\(b'_j,\beta'_j;B'_j)_{1,M},(b'_j,\beta'_j)_{M+1,Q}\end{matrix}\right.\right] \bar{H}^{m,n}_{p,q}\left[zX^{\rho}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\frac{\sqrt{\pi}}{2a(4ab+c)^{\eta+\frac{1}{2}}}\sum_{k_1}^{n_1/m_1}\cdots\sum_{k_r}^{n_r/m_r}\left[(-n_1)_{m_1k_1}\cdots(-n_r)_{m_rk_r}\right]$$

$$A(k_1,\ldots,k_r)\frac{c_1^{k_1}}{(4ab+c)^{k_1\delta_1}\,k_1!}\cdots\frac{c_r^{k_r}}{(4ab+c)^{k_r\delta_r}\,k_r!}\frac{1}{2\pi i}\int_L \bar{f}(\xi)$$

$$\bar{H}^{m+1,n}_{p+1,q}\left[z(4ab+c)^{\rho}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p},\left\{-\eta+\sigma\xi+\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i);\rho;1\right\}\\\left\{\frac{1}{2}-\eta+\sigma\xi+\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i);\rho;1\right\},(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]$$

$$(4ab+c)^{\xi\sigma}d\xi \qquad (10)$$

The above result will be converging under the following conditions: i. $a>;b\geq 0;c+4ab>0$ and $\eta>,\delta_i\geq 0,\sigma>0,\rho\geq 0$, ii. $-\eta+\sigma\min_{1\leq j\leq M}\mathrm{Re}(\frac{b'_j}{\beta'_j})+\rho\min_{1\leq j\leq m}\mathrm{Re}(\frac{b_j}{\beta_j})<\frac{1}{2}$, iii. $|\arg z|<\frac{1}{2}\pi\Omega$, where Ω is given by equation (4)

Second Integral Formula:

$$\int_0^\infty \frac{1}{x^2}X^{-\eta-1} S^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}\left[c_1X^{\delta_1},\ldots,c_rX^{\delta_r}\right] \bar{H}^{m,n}_{p,q}\left[zX^{\rho}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\frac{\sqrt{\pi}}{2b(4ab+c)^{\eta+\frac{1}{2}}}\sum_{k_1}^{n_1/m_1}\cdots\sum_{k_r}^{n_r/m_r}\left[(-n_1)_{m_1k_1}\cdots(-n_r)_{m_rk_r}\right]$$

$$A(k_1,\ldots,k_r)\frac{c_1^{k_1}}{(4ab+c)^{k_1\delta_1}\,k_1!}\cdots\frac{c_r^{k_r}}{(4ab+c)^{k_r\delta_r}\,k_r!}\frac{1}{2\pi i}\int_L \bar{f}(\xi)$$

$$\bar{H}^{m+1,n}_{p+1,q}\left[z(4ab+c)^{\rho}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p},\left\{-\eta+\sigma\xi+\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i);\rho;1\right\}\\\left\{\frac{1}{2}-\eta+\sigma\xi+\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i);\rho;1\right\},(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]$$

$$(4ab+c)^{\xi\sigma}d\xi \qquad (11)$$

The above result will be converging under the following conditions: i. $a\geq 0;b>0;c+4ab>0$ and $\eta>,\delta_i\geq 0,\sigma>0,\rho\geq 0$, ii. $-\eta+\sigma\min_{1\leq j\leq M}\mathrm{Re}(\frac{b'_j}{\beta'_j})+\rho\min_{1\leq j\leq m}\mathrm{Re}(\frac{b_j}{\beta_j})<\frac{1}{2}$, iii. $|\arg z|<\frac{1}{2}\pi\Omega$, where Ω is given by equation (4)

Third Integral Formula:

$$\int_0^\infty (a+\frac{b}{x^2})X^{-\eta-1} S^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}\left[c_1X^{\delta_1},\ldots,c_rX^{\delta_r}\right] \bar{H}^{M,N}_{P,Q}\left[X^{\sigma}\left|\begin{matrix}(a'_j,\alpha'_j;A'_j)_{1,N},(a'_j,\alpha'_j)_{N+1,P}\\(b'_j,\beta'_j;B'_j)_{1,M},(b'_j,\beta'_j)_{M+1,Q}\end{matrix}\right.\right] \bar{H}^{m,n}_{p,q}\left[zX^{\rho}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\frac{\sqrt{\pi}}{(4ab+c)^{\eta+\frac{1}{2}}}\sum_{k_1}^{n_1/m_1}\cdots\sum_{k_r}^{n_r/m_r}\left[(-n_1)_{m_1k_1}\cdots(-n_r)_{m_rk_r}\right]$$

$$A(k_1,\ldots,k_r)\frac{c_1^{k_1}}{(4ab+c)^{k_1\delta_1}\,k_1!}\cdots\frac{c_r^{k_r}}{(4ab+c)^{k_r\delta_r}\,k_r!}\frac{1}{2\pi i}\int_L \bar{f}(\xi)$$

$$\bar{H}^{m+1,n}_{p+1,q}\left[z(4ab+c)^{\rho}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p},\left\{-\eta+\sigma\xi+\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i);\rho;1\right\}\\\left\{\frac{1}{2}-\eta+\sigma\xi+\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i);\rho;1\right\},(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]$$

$$(4ab+c)^{\xi\sigma}d\xi \qquad (12)$$

The above result will be converging under the following conditions: i. $a>0; b>0; c+4ab>0$ and $\eta>, \delta_i \geq 0, \sigma>0, \rho \geq 0$, ii. $-\eta+\sigma \min_{1\leq j\leq M} \mathrm{Re}(\frac{b_j'}{\beta_j'}) + \rho \min_{1\leq j\leq m} \mathrm{Re}(\frac{b_j}{\beta_j}) < \frac{1}{2}$, iii. $|\arg z| < \frac{1}{2}\pi\Omega$, where Ω is given by equation (4).

Proof: To prove the first integral, we first express $\overline{H}$ -function occurring on the L.H.S. of equation (10) in terms of Mellin-Barnes type of contour integral given by equation (1) and general class of multivariable polynomials $S_{n_1,n_2,\ldots,n_r}^{m_1,m_2,\ldots,m_r}[x_1,\ldots,x_r]$ in series form with the help of (5) and then interchanging the order of integration and summation.
We get =

$$\sum_{k_1}^{n_1/m_1} \cdots \sum_{k_r}^{n_r/m_r} \left[(-n_1)_{m_1k_1} \cdots (-n_r)_{m_rk_r} \right] A(k_1,\ldots,k_r) \frac{c_1^{k_1}}{(4ab+c)^{k_1\delta_1} k_1!} \cdots \frac{c_r^{k_r}}{(4ab+c)^{k_r\delta_r} k_r!} \frac{1}{2\pi i}\int_L \bar{f}(\xi) \frac{1}{2\pi i}\int_L \bar{\psi}(\zeta) \left[\int_0^\infty \left[(ax+\frac{b}{x})^2 + c \right]^{-\eta+\sigma\xi+\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i)-1} dx \right] d\xi\, d\zeta \qquad (13)$$

Further using the result (7) the above integral becomes=

$$\sum_{k_1}^{n_1/m_1} \cdots \sum_{k_r}^{n_r/m_r} \left[(-n_1)_{m_1k_1} \cdots (-n_r)_{m_rk_r} \right] A(k_1,\ldots,k_r) \frac{c_1^{k_1}}{(4ab+c)^{k_1\delta_1} k_1!} \cdots \frac{c_r^{k_r}}{(4ab+c)^{k_r\delta_r} k_r!} \frac{1}{2\pi i}\int_L \bar{f}(\xi) \frac{1}{2\pi i}\int_L \bar{\psi}(\zeta)$$

$$\frac{1}{2\pi i}\int_L \bar{\psi}(\varsigma) Z^{\varsigma} \left\{ \frac{\sqrt{\pi}}{2a(4ab+c)^{\eta-\sigma\xi-\rho\varsigma-\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i)+\frac{1}{2}}} \frac{\Gamma\left\{\eta-\sigma\xi-\rho\varsigma-\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i)+\frac{1}{2}\right\}}{\Gamma\left\{\eta-\sigma\xi-\rho\varsigma-\sum_{k_r}^{n_r/m_r}\prod_{i=1}^{r}(k_i\delta_i)+1\right\}} \right\} d\varsigma\, d\xi$$

Then interpreting with the help of (1) and (14) provides first integral. Proceeding on the same parallel lines, integral second and third given by equation (11) and (12) can be easily obtained by using the results (8) and (9) respectively.

Special Cases

1. If we are choosing in equation (10), $n_i = 1$; where $n_i = 1, 2 \ldots; \forall i \in 1, 2, \ldots, r$, Then

$$\int_0^\infty X^{-\eta-1} S_L^{m_1,\ldots,m_r}\left[c_1X^{\delta_1},\ldots,c_rX^{\delta_r}\right] \overline{H}_{P,Q}^{M,N}\left[X^{\sigma}\left|\begin{matrix}(a_j',\alpha_j';A_j')_{1,N},(a_j',\alpha_j')_{N+1,P}\\(b_j',\beta_j';B_j')_{1,M},(b_j',\beta_j')_{M+1,Q}\end{matrix}\right.\right]$$

$$\overline{H}_{p,q}^{m,n}\left[zX^{\rho}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$= \frac{\sqrt{\pi}}{2a(4ab+c)^{\eta+\frac{1}{2}}} \sum_{k_1,\ldots,k_r=0}^{h_1k_1,\ldots,h_rk_r \leq L} (-L)_{h_1k_1\ldots h_rk_r} A(L;k_1,\ldots,k_r)$$

$$\frac{c_1^{k_1}}{(4ab+c)^{k_1\delta_1}\lfloor k_1} \cdots \frac{c_r^{k_r}}{(4ab+c)^{k_r\delta_r}\lfloor k_r} \frac{1}{2\pi i}\int_L \bar{f}(\xi)$$

$$\overline{H}_{p+1,q}^{m+1,n}\left[z(4ab+c)^{\rho}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p},\left\{-\eta+\sigma\xi+\sum_{i=0}^{r}k_i\delta_i,\rho;1\right\}\\\left\{\frac{1}{2}-\eta+\sigma\xi+\sum_{i=0}^{r}k_i\delta_i,\rho;1\right\},(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]$$

$$(4ab+c)^{\xi\sigma} d\xi \quad (15)$$

Similarly If we are choosing in equation (11), (12) put $n_i = 1$; where $n_i = 1, 2 \ldots; \forall i \in 1, 2, \ldots, r$, Then we get new and unknown results.

Further If we put, $A_J' = B_J' = A_J = B_J = 1$, $\alpha_J' = \beta_J' = \alpha_J = \beta_J = 1$ then the $\overline{H}$ –function reduces to general type of G-function[6], which is also the new special case in[10].

Conclusion

The present paper is to evaluate a new unified integrals whose function involved in the integral formulae as well as their arguments are quite general in nature and so our findings provide interesting unifications and extensions of a number of (known and new) results.

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