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Tietze's Extension Theorem in Soft Topological Spaces

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Abstract

Human inventions that made swift and remarkable progress in global human life ranging from the inventions of fire from stone age to the artificial intelligence of the current period. But this sophisticated turning of developments increased the uncertainties of day-to-day life which demands more parametrization tools in dealing with this problems.Inspired from Pawlak's work on Hard and Soft sets in the proceeding of "EWorkshop on rough sets in 1993", Molodtsov introduced in 1999 a new theory to soften all the existing difficulties which is known as "soft set theory"[1,2,3]. In this paper it's going to prove Tietze's extension theorem in soft topological spaces.

Keywords: Soft topology, soft open set, soft closed set, soft neighborhood, soft normal space, soft continuity.

Introduction

Using Urysohn's lemma in soft topological space it's proved that real –valued soft mapping on it's soft subspace admit an extension to the whole space[4]. This is given by the Tietze Extension Theorem in soft topological space[1-4].

Preliminaries

Definition:1 Let $\widetilde{U}$ be the initial universe, E the set of parameters, A the non-empty subset of E and P($\widetilde{U}$) the power set of $\widetilde{U}$. The pair (R,A) is called the *softset* over $\widetilde{U}$ where R is the mapping given by R : A → P($\widetilde{U}$)[2,3,5] or R : A →$2^{P(\widetilde{U})}$.[6]

Remark:1 R(e) is the e-approximate element[5,7].

Remark:2 In null soft set ($\widetilde{\emptyset}$,A) each e-approximate element of the attribute set is null set and in absolute soft set ($\widetilde{U}$,A) it is the universal set [8].

Remark: 3 Union of two soft sets (R,A) and (S,B) over common universal set $\widetilde{U}$ is (Z,P) where P=A∪B and every e-approximate element of P is either "e-approximate element of R"or "S" or "union of the e-approximate elements of both" when the attribute set takes values "from A but not from B"or "from B but not from A" or "from the intersection of both"-respectively[8].

Remark: 4 Intersection of (R,A) and (S,B) is the e-approximate element of the intersection of both of the attribute sets which will either the e-approximate element of one or the other[8]

Example: 1 In a wedding ceremony, on the basis of a special request from the friends and relatives, marrying couple is compelled to choose their preference of wedding hall from the followings options : i. stylish hall, ii. costly hall, iii. cheap hall iv. attractive hall v. luxurious hall, vi. spacious hall, vii. hall in a prime area, viii. A/C hall, ix. a hall with less accommodating capacity. Couple's choice is an A/C hall with less accommodating capacity. A search results out that "Regency hall" and "Tulips hall" are only available at the time of wedding. Soft set theory extends its tools to this life situation by taking the parameter set as E ={e_1= stylish hall, e_2 = costly hall, e_3 = cheap hall, e_4 = attractive hall, e_5 = luxurious hall, e_6 = spacious hall, e_7 = hall in a prime area, e_8 = A/C hall, e_9 = a hall with less accommodating capacity}, universal set as $\widetilde{U}$ ={h_1 = Regency hall, h_2 = Tulips hall}, attribute set under consideration on the present context as A={e_8,e_9}, power set as P($\widetilde{U}$) ={$\widetilde{\emptyset}$,{h_1},{h_2},{h_1, h_2}} = {$\widetilde{\emptyset}$,{h_1},{h_2},$\widetilde{U}$} and corresponding mapping as R:{e_8,e_9} {$\widetilde{\emptyset}$,{h_1},{h_2},$\widetilde{U}$}. Soft sets we got here are : (R_0,A), (R_1,A), (R_2,A), (R_3,A),(R_4,A), (R_5.A), (R_6,A), (R_7.A), (R_8,A), (R_9,A), (R_{10},A), (R_{11},A), (R_{12},A), (R_{13},A), (R_{14},A), (R_{15},A). Among these (R_5,A), (R_6,A),(R_7,A), (R_9,A), (R_{10},A), (R_{11},A), (R_{13},A), (R_{14},A), (R_{15},A) are the soft sets which meets the couple's requirement. If we collect the information about "the degree of importance given by the couple for the considering attributes" then it could be used to produce the exact result in this life situation with one more filtration (using fuzzy soft set theory).

Remark: 5 Shabir and Naz in 2011 introduced soft topological spaces by defining soft topology on the collection τ of soft sets over $\widetilde{U}$ but at the same time Cagman et.al started studing in this direction independently in a more general way by introducing soft topology on a soft set[9,10].

Definition: 2 The collection τ of soft sets over $\widetilde{U}$ is a soft topology over $\widetilde{U}$ if : i. null and absolute soft sets belongs to τ, ii. arbitrary union of soft sets in τ belongs to τ, iii. the intersection of any two soft sets in τ belongs to τ.[10]

Remark: 6 In ordinary topology the pair ($\widetilde{U}$, τ) denotes a topological space while in soft topology the triplet ($\widetilde{U}$,A, τ) denotes a *soft topological space* over $\widetilde{U}$ [10]

Definition: 3 In a soft topological space members of τ are called soft open sets and their relative complements soft closed sets[10].

Remark: 7 The elements of soft topology are ordered-pairs but in general topology it is not ! - which turns out to be the high-lighting difference between general and soft topologies[11].

Remark: 8 An ordinary topological space can be considered as soft topological space[12] [since every ordinary topological space can be expressed as fuzzy soft topological space and a fuzzy topological space is a special case of soft topological space[2,13]] but the converse need not be true[12].

Remark: 9 If (C, A) is soft closed then it's relative complement should be soft open which in turn the member of some soft topology.

Remark: 10 Fixing the attribute set guarantees the accuracy and efficiency. The family of soft sets denoted as SS($\widetilde{U}$,A)[14] or SS$(\widetilde{U})_A$ gives those soft sets in which parameter set is same[15]. But the set of all soft sets over $\widetilde{U}$ is denoted by S($\widetilde{U}$) in which parameter set need not be same [14,15].

Example: 2 Suppose (R,A) and (R,B) are two soft sets over $\widetilde{U}$. Then (R,A)$\widetilde{\in}$ SS$(\widetilde{U})_A$ "but (R,B)$\widetilde{\in}$ SS$(\widetilde{U})_A$ only if A=B"or (R,B)$\widetilde{\in}$ SS$(\widetilde{U})_B$ "but (R,A)$\widetilde{\in}$ SS$(\widetilde{U})_B$ only if B=A"[14,15].

Definition: 4 (R,A) which is obviously a member of SS$(\widetilde{U})_A$ is a soft point of the universal set (Ũ,A) denoted by $e_{(R,A)}$ or e_R, if the mapping R carry each and every element of the attribute set to non-empty elements of P($\widetilde{U}$) and the left elements mapped to empty set [14,15]. [since A⊂ E there would be elements left unless A=E. Here e$\widetilde{\in}$ A then R(e)≠ $\emptyset$ and R(e)= $\emptyset$, ∀ e $\widetilde{\in}$ (E – A)]

Definition: 5 The soft set (R,A) is a soft point of some soft set namely (H,A) denoted by $e_{(R,A)}\widetilde{\in}$ (H,A), if for the element e $\widetilde{\in}$ A and R(e) $\widetilde{\subseteq}$ H(e).

Proposition: 1 Let $e_{(R,A)}$ $\widetilde{\in}$ ($\widetilde{U}$,A) and (H,A) $\widetilde{\subseteq}$ ($\widetilde{U}$,A).If $e_{(R,A)}$ $\widetilde{\in}$ (H,A),then $e_{(R,A)}$ $\widetilde{\notin}$ $(H,A)^C$ [16].

Definition: 6 Suppose (H,A) be the soft set over ($\widetilde{U}$,A, τ). The soft set $e_{(R,A)}$in the absolute soft set is called a soft interior point of the soft set (H,A) if there exists a soft open set (K,A) such that $e_{(R,A)}$ $\widetilde{\in}$ (K,A) $\widetilde{\subseteq}$ (H,A)[5].Here (H,A) is called the soft neighborhood of the soft point $e_{(R,A)}$.

Proposition: 2 If $e_{(R,A)}$ $\widetilde{\in}$ ($\widetilde{U}$,A) for all e∈ A and (H, A) be a soft open set in a soft topological space ($\widetilde{U}$, A, τ) then every soft point $e_{(R,A)}$ is a soft interior point .

Definition: 7 (H,A) in a soft topological space ($\widetilde{U}$,A, τ) is called the soft neighborhood of (R,A) if there exists a soft open set (K,A) such that (R,A) $\widetilde{\subseteq}$ (K,A) $\widetilde{\subseteq}$ (H,A). The neighborhood system of the soft point $e_{(R,A)}$ is the family of all its neighborhoods denoted by N τ ($e_{(R,A)}$)[16].

Theorem: 1 The necessary and sufficient condition for a soft set to be a soft neighborhood of each of it's soft elements is that, it is "soft open".

Definition: 8 If there exist two disjoint soft open sets (T_1,A) and (T_2,A) containing any two non-empty disjoint soft closed sets (C,A) and (D,A) respectively in a soft topological space ($\widetilde{U}$,A, τ) then that space is called a *soft normal space*[10].

Definition: 9 Let ($\widetilde{U}$,A, τ) be a soft topological space over $\widetilde{U}$. Let (R,A) be any soft open set over $\widetilde{U}$ and $\widetilde{V}$ be a non-empty subset of $\widetilde{U}$ Then $\tau_{\widetilde{V}}$ ={($\widetilde{V}$R,A) /(R,A)∈ τ }is said to be soft relative topology on $\widetilde{V}$and ($\widetilde{V}$,A,$\tau_{\widetilde{V}}$) is called a *soft subspace* of ($\widetilde{U}$,A, τ)[10].

Proposition: 3 Let ($\widetilde{U}$,A, τ)be a soft topological space over $\widetilde{U}$ and $\widetilde{V}$ be a non-empty subset of $\widetilde{U}$.Then ($\widetilde{V}$,$\tau_{\alpha_{\widetilde{V}}}$) is a subspace of ($\widetilde{U}$,$\tau_\alpha$) for each $\alpha \in$ A.[10]

Definition: 10 Let ($\widetilde{U}$,A, τ) and ($\widetilde{W}$,A, τ') be two soft topological spaces g: ($\widetilde{U}$,A, τ)→($\widetilde{W}$,A, τ') be a mapping. For each soft neighborhood (H,A) of (g(e_x),A), if there exists a soft neighborhood (G,A) of (e_x, A) such that g ((G,A))⊂(H,A). Then g is said to be soft continuous mapping at (e_x , A).If g is soft continuous mapping for all (e_x, A), g is called *soft continuous mapping*[17].

Lemma: 1(Urysohn's lemma in soft topological spaces) Let ($\widetilde{U}$,A, τ) be a soft normal space; let (E,A) and (I,A) be disjoint soft closed subsets of ($\widetilde{U}$,A, τ). Let [a,b] be a closed interval in the real line. Then there exists a soft continuous map r: ($\widetilde{U}$,A, τ)⟶[a,b] such that r(e_x) = a ∀ e_x $\widetilde{\in}$ (E,A) , b ∀ e_x $\widetilde{\in}$ (I,A). Converse holds[4].

Theorem: 1 (Tietze' extension theorem in soft topological spaces) If ($\widetilde{U}$,A, τ) is soft normal then for the soft continuous real valued function f defined on it's every soft closed subspace (C,A) admits a soft continuous extension g: ($\widetilde{U}$,A, τ) ⟶[a,b].

Proof Assume that ($\widetilde{U}$,A, τ) is soft normal. Let f : (C,A) ⟶ [a,b] be a soft continuous map, (C,A)being a soft closed subspace of ($\widetilde{U}$,A, τ).Take a=-1 and b=1. Define a soft map f_0 : (C,A) ⟶[-1,1] such that $f_0(e_x)$ = $f(e_x)$ ∀ $e_x$$\widetilde{\in}$ (C,A). Suppose

(M_0,A) and (H_0,A) are two non-empty disjoint soft subsets of (C,A) such that $(M_0,A) = \{e_x / f_0(e_x) \leq -1/3\}$, $(H_0,A)=\{e_x / f_0(e_x) \geq 1/3\}$. Since (C,A) is soft closed in $(\widetilde{U},A,\tau)$, they are soft closed in $(\widetilde{U},A,\tau)$also. Applying lemma1 there exists a soft continuous function $g_0:(\widetilde{U},A,\tau) \rightarrow [-1/3,1/3]$ such that $g_0(M_0,A) = -1/3$ and $g_0(H_0,A) = 1/3$. Write $f_1=(f_0-g_0)$.Then $|f_1(e_x)| = |(f_0-g_0)e_x| = |f_0(e_x) - g_0(e_x)| \leq 2/3$. Let $(M_1,A) = \{e_x / f_1(e_x) \leq (-1/3).(2/3) = (-2/9)\}$ and $(H_1,A) = \{e_x / f_1(e_x) \geq (1/3).(2/3)=(2/9)\}$. Then (M_1,A) and (H_1,A) are non-empty disjoint soft closed sets in $(\widetilde{U},A,\tau)$ and hence by lemma 1 there exists a soft continuous function such that $g:(\widetilde{U},A,\tau) \rightarrow [-2/9, 2/9]$ such that $g_1(M_1,A) = -2/9$ and $g_1(H_1,A) = 2/9$. Again we define a soft function f_2 on (C,A) such that $f_2=(f_1-g_1) = (f_0-g_0-g_1) = f_0-(g_0+g_1)$. Then$|f_2(e_x)|=| f_0(e_x) - (g_0+g_1)(e_x)| \leq (2/3)^2$. Continuing this process we get a sequence of soft functions $\langle f_0,f_1,f_2,\ldots\ldots f_n,\ldots\ldots \rangle$ defined on (C,A) such that $|f_n(e_x)| \leq (2/3)^n$ and a sequence $\langle g_0,g_1,g_2,\ldots.. \rangle$ defined on $(\widetilde{U},A,\tau)$ such that $|g_n(e_x)| \leq (1/3)(2/3)^n$ and $f_n=f_0-(g_0+g_1+\ldots\ldots\ldots+g_{n-1})$. Let $S_n = \sum_{r=0}^{n-1} g_r$. Now S_n can be regarded as partial sums of bounded soft continuous functions defined on $(\widetilde{U},A,\tau)$. Since the space of bounded soft continuous real valued functions is complete and $|g_n(e_x)| \leq 1/2.(2/3)^n$ and $\sum_{n=0}^{\infty} 1/3(2/3)^n = 1$, the sequence $\langle S_n \rangle$ converges uniformly on $(\widetilde{U},A,\tau)$ to g (say) when $|g(e_x)| \leq 1$. $|f_n(x)| \leq (2/3)^n \Rightarrow \langle S_n \rangle$ converges uniformly on $(\widetilde{U},A,\tau)$ to f_0 = f, say. Hence g = f on C. Thus g is a soft continuous extension of f to $(\widetilde{U},A,\tau)$ which satisfies the given conditions.

Converse of Tietze's extension theorem in soft topological spaces : Let $(\widetilde{U},A,\tau)$ be a soft topological space such that every soft continuous real-valued function $f:(C,A) \rightarrow [a,b]$ has a soft continuous extended function $g:(\widetilde{U},A,\tau) \rightarrow [a,b]$. Here (C,A) is a soft closed subspace of $(\widetilde{U},A,\tau)$ and [a,b] being a real closed interval. Let (F_1,A) and (F_2,A) be two soft closed disjoint subsets of $(\widetilde{U},A,\tau)$. Define a soft map $f : (F_1,A)\widetilde{\cup}(F_2,A) \rightarrow [a,b]$ such that $f(e_x) = a$ if $e_x \widetilde{\in} (F_1,A)$ and b if $e_x \widetilde{\in} (F_2,A)$. This soft map is continuous over soft subspace $(F_1,A)\widetilde{\cup}(F_2,A)$. By assumption f can be extended to a soft continuous map $g : (\widetilde{U},A,\tau) \rightarrow [a,b]$ such that $g(e_x) = a$ if $e_x \widetilde{\in} (F_1,A)$ and b if $e_x \widetilde{\in} (F_2,A)$.This map g satisfies Urysohn's lemma in soft topological spaces[4] and hence $(\widetilde{U},A,\tau)$ is soft normal.

Remark: 11 Tietze's extension theorem in soft topological spaces along with it's converse is known as Tietze's characterization of soft normality in soft topological spaces

Theorem: 2 Let (J,A) be a soft closed subset of a soft normal space $(\widetilde{U},A,\tau)$ and suppose $t : (C,A) \rightarrow (-1,1)$ is soft continuous. Then there exists a soft continuous function $T: (\widetilde{U},A,\tau) \rightarrow (-1,1)$ such that$T(e_x)= t(e_x)$ for all $e_x \in$ (J,A).

Proof Define $p(e_x)= t(e_x)/[1+t(e_x)]$ for $e_x \in$ (J,A). Then p is soft continuous and takes values in (-1/2,1/2). So by the theorem above, it has an extension, $P:(\widetilde{U},A,\tau) \rightarrow [-1/2,1/2]$. Now let $(P,A)=\{e_x \in (\widetilde{U},A,\tau):P(e_x) = ½$ or $P(e_x) = -1/2\}$. Then (P,A) is a soft closed subset of $(\widetilde{U},A,\tau)$ since $(P,A)= P^{-1}(\{-1/2,1/2\})$. Here $(J,A)\widetilde{\cap}(P,A) = (\widetilde{\emptyset},A)$, for if $e_x \in$ (J,A) then $P(e_x) = p(e_x) = t(e_x)/[1+|t(e_x)|]$ and $P(e_x) = \pm 1/2$ would mean $2|t(e_x)| = 1+|t(e_x)|$, a contradiction. So by Urysohn's lemma there exists a soft continuous function $q : (\widetilde{U},A,\tau) \rightarrow [0,1]$ which equals 0 on (P,A) and 1on (J,A). Define $Q(e_x) = q(e_x) P(e_x)$ for $e_x \in (\widetilde{U},A,\tau)$. Then Q is soft continuous and takes values only in (-1/2,1/2). Now define $T:(\widetilde{U},A,\tau) \rightarrow R$ by $T(e_x) = Q(e_x)/[1-|Q(e_x)|]$ for $e_x \in (\widetilde{U},A,\tau)$. T is a well defined soft continuous function on $(\widetilde{U},A,\tau)$ on $(\widetilde{U},A,\tau)$ and it takes values in (-1,1) as Q takes values in (-1/2,1/2). So we regard T as a soft continuous function from $(\widetilde{U},A,\tau)$ into (-1,1). It remains to verify that T is an extension of t. Let $e_x \in$ (J,A). If $t(e_x) \geq 0$ then $Q(e_x) = q(e_x)P(e_x)=P(e_x) = p(e_x) = \{t(e_x)/[1+t(e_x)]\} \geq 0$ and so $T(e_x) = t(e_x)$ by direct calculation of $T(e_x)$. Similarly $t(e_x) = T(e_x)$ when $t(e_x) \leq 0$.This shows that t has T as an extension.

Corollary: 1 Any soft continuous real-valued function on a soft closed subset of a soft normal space can be extended soft continuously to the whole space.

Proof: Any open interval and the real-line are homeomorphic to (-1,1) so the corollary follows.

Conclusion

Topology is the mathematical study of properties of objects that are preserved through deformations, twisting and stretching without tearing of objects and it can be also used for decision making. One of the basic problems of topology is to determine whether two given topological spaces are homeomorphic or not! Continuity plays a vital role in this sence. In this paper we have proved that Tietze's extension theorem is true in soft topology also using Urysohn's lemma in soft topological spaces. Extension of Soft continuity is the main concept behind Tietze's extension theorem in soft topological spaces.

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