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Survey on Intersection of two Maximum Length Paths in Connected Graph

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Abstract

In the last decade important results on intersection of two maximum length paths in connected graph were discovered. The aim of this paper is to discuss in detail the progress on the problems of intersecting two maximum length paths, providing some new result in the process. Furthermore we establish the results concerning the intersection of two maximum length paths in 2-connected graph and 3-connected graph.

Keywords: Connected graph, order of path, maximum length path.

Introduction

The intersection of two maximum length paths have been studied in detail and Zamfirescu gave a short survey[1]. It is well known and straightforward to verify in a connected graph any two maximum length paths share a common vertex. However considering the intersection of more than 2 longest paths gets more fascinating.

In general it is still unknown whether any three maximum length paths of every connected graph share a common vertex, but Axenovich proved it for outerplanar graphs[2]. De-Rezende, Martin and Wakabayashi proved in a connected graph if all non-trivial blocks are Hamiltonian then any three maximum length path of the graph share a vertex[3]. Even though it seems as if the property of having a vertex meet to all maximum length paths is too strong, there are some classes of graphs for which this property holds.

Zamfirescu found a graph with 12 vertices in which intersection of all longest path is empty[1]. Skupien obtained, For $p \geq 7$, a connected graph in which some p longest paths have empty intersection, but every p-1 longest paths have a common vertex[4]. Balister et al. showed that in a circular arc graph all longest path share a common vertex[5]. Moreover Klavžar and Petkovšek proved that if a graph is a split graph, the intersection of all longest path is non-empty[6]. In this paper we will explore what happens when we look at 2-*connected* graphs and 3-*connected* graphs, leading us to make a conjecture about the intersection of any two maximum length paths.

Definitions and Results

We consider undirected graph G(V,E), where V and E are the set of vertices and edges respectively. A **path** in a graph is a sequence of vertices in which each vertex is adjacent to the next one. Edges can be part of a path only once. A graph G is connected if for every pair of vertices u and v in V, there exist a path in G with u and v as start and end points. Define the order of path P is the number of vertices in path $P(v_1,v_2,\ldots,v_k)$ and v_1, v_k are the terminal vertices of path P. By $l^*(P)$, we denote the order of path in G. The path with maximum order is said to be maximum length path.

Theorem 2.1(Pigeonhole Principle): Consider there are n objects and m places. If $n \geq (p-1)m+1$ then at least one place must have p objects in it.

Theorem 2.2: If G is a connected graph then any two maximum length paths share a common vertex.

Theorem 2.3: If G is a connected graph and intersection of any two maximum length paths is exactly one common vertex then this common vertex lies exactly in the centre of the longest path.

Main results

Theorem 3.1:- If G is a connected graph and intersection of any two maximum length paths is exactly the one common vertex, then the order of path to the common vertex must be same from all the four terminal vertices of two paths.

Proof:- Suppose $P_1[u_1,u_n]$ and $P_2[v_1,v_n]$ be any two maximum length paths in Graph G such that $V(P_1)\cap V(P_2)=\{x\}$ and let P_1^* be the path from u_1 to x , P_2^* be the path from x to u_n , P_3^* be the path from v_1 to x and P_4^* be the path from x to v_n . Assume $l^*(P_i^*)\neq l^*(P_j^*)\ \forall\ i\neq j,\ 1\leq i\leq 4,\ 1\leq j\leq 4$.

Then $l^*(P_1^*) + l^*(P_2^*) = n$ and $l^*(P_3^*)+ l^*(P_4^*) = n$

Without loss of generality, assume that P_1^* is longest this means P_2^* must be shortest. By the Pigeonhole Principle, we know that $l^*(P_3^*)$ and $l^*(P_4^*)$ cannot both equal $l^*(P_2^*)$, and that neither can be less than $l^*(P_2^*)$. Since this would contradict that P_1^* was the longest.

Now consider the paths $P_1^*[v_1,x]P_3^*[x,u_1]$ and $P_1^*[v_1,x]P_4^*[x,u_n]$. Again by the Pigeonhole Principle, one of these paths must be longer than n. So our assumption is false and hence $l^*(P_i^*)=l^*(P_j^*)$, $\forall$ $i\neq j$, $1\leq i\leq 4$, $1\leq j\leq 4$.

Corollary 3.1: If G is a connected graph and intersection of any two maximum length paths is exactly one common vertex x such that m is the length from x to terminal vertex of maximum length path then order of longest path is 2m.

Proof: Consider m is the length from x to terminal vertex of maximum length path $P_1[u_1,u_n]$ and $P_2[v_1,v_n]$. Let P_1^* be the path from u_1 to x, P_2^* be the path from x to u_n , P_3^* be the path from v_1 to x and P_4^* be the path from x to v_n then by theorem 3.1, $l^*(P_i^*) = l^*(P_j^*) = m$ $\forall$ $i\neq j$, $1\leq i\leq 4$, $1\leq j\leq 4$
$\Rightarrow l^*(P_1)= l^*(P_1^*)+ l^*(P_2^*)=2m$
Hence the lemma is proved.

A graph G(V,E) is *k-connected* if $|G|>k$ and G-X is connected for every set $X\subseteq V$ and $|X|<k$. In this part we will look at 2-*connected* graph and 3-*connected* graph and determine how many vertices must the two maximum length paths share.

Theorem 3.2: If G is a 2-*connected* graph with r vertices then the intersection of any two maximum length paths is m vertices, where $2\leq m < r$

Proof:- Consider $P_1^*[u_1,u_{2n+1}]$ and $P_2^*[v_1,v_{2n+1}]$ be any two maximum length paths in G, then by theorem 2.2, they must share at least one vertex. Denote this vertex by *x*, then by theorem 2.3, *x* must lies in the centre and therefore $x=v_{n+1}=u_{n+1}$. Since G is *2-connected* there must be another path between P_1^* and P_2^*, denote it by P, meeting P_1^* at vertex say c and meeting P_2^* at vertex say d.

Consider another two paths (figure-1)
$P_1=P_1^*[u_1,c]P[c,d]P_2^*[d,v_{2n+1}]$
$P_2=P_2^*[v_1,d]P[d,c]P_1^*[c,u_{2n+1}]$

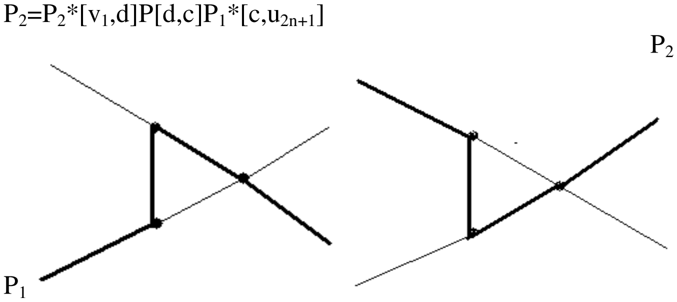

Figure-1

$l^*(P_1)+ l^*(P_2) = l^*(P_1^*[u_1,c]P[c,d]P_2^*[d,v_{2n+1}]) + l^*(P_2^*[v_1,d]P[d,c]P_1^*[c,u_{2n+1}])$
$= l^*(P_1^*[u_1,c]\ P_1^*[c,u_{2n+1}]) + l^*(P_2^*[v_1,d]\ P_2^*[d,v_{2n+1}]) + 2\,l^*(P[c,d])$
$= l^*(P_1^*) + l^*(P_2^*) + 2l^*(P)$

So by the Pigeonhole principle, one of the paths P_1 or P_2 must be longer than P_1^* and P_2^*. So our assumption is false and hence the intersection of any two maximum length paths contains m vertices.

Theorem 3.3: If G is a 3-*connected* graph with r vertices then the intersection of any two maximum length paths contains m vertices, where $3\leq m < r$.

Proof: Let $P_1^*[u_1, u_n]$ and $P_2^*[v_1, v_n]$ be the maximum length paths in graph G. Clearly G is 2-*connected* so by theorem 3.2, P_1^* and P_2^* have at least two vertices in common, namely $a=u_i=v_i$ and $b=u_j=v_j$ where i<j. Assume that $i\leq j-2$. By removing a and b from G we see that we get six disjoint pieces and since this graph G is 3-*connected*, there must be a path in G connecting them. Particularly, connecting the path from $[u_{i+1},u_{j-1}]$ to $[u_1,u_{i-1}]$ or $[u_{j+1},u_n]$ or $[v_1,v_{i-1}]$ or $[v_{j+1},v_n]$. Let this path be L and let it connect at vertex d in $[u_{i+1},u_{j-1}]$ and vertex c in $[u_1,u_{i-1}]$ or $[u_{j+1},u_n]$ or $[v_1,v_{i-1}]$ or $[v_{j+1},v_n]$. Without loss of generality assume that c is on the path between u_1 and a and d is on P_1^* between a and b.

Now look at the two paths (figure-2)
$P_1=P_1^*[u_1,c]L[c,d]P_1^*[d,a]P_2^*[a,v_n]$
$P_2=P_2^*[v_1,a]P_1^*[a,c]L[c,d]P_1^*[d,u_n]$

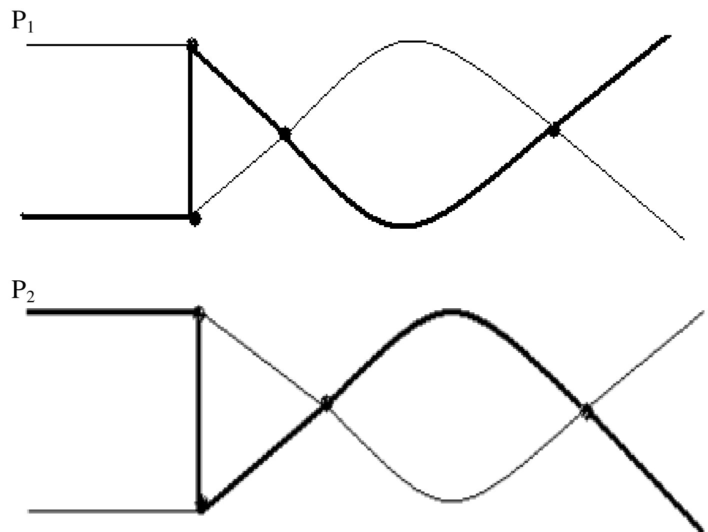

Figure-2

Then
$l^*(P_1)+l^*(P_2)=l^*(P_1^*[u_1,c]L[c,d]P_1^*[d,a]P_2^*[a,v_n])+l^*(P_2^*[v_1,a]P_1^*[a,c]L[c,d]P_1^*[d,u_n])$
$=l^*(P_1^*[u_1,c]P_1^*[d,a]P_1^*[a,c]P_1^*[d,u_n])+l^*(P_2^*[v_1,a]P_2^*[a,v_n])+2l^*(L[c,d])$
$= l^*(P_1^*) + l^*(P_2^*) + 2l^*(L)$
So by the Pigeonhole principle, one of the paths P_1 or P_2 must be longer than P_1^* and P_2^*.

If i+1=j, then we can replace a and b with z such that $M(z)=\{v_{i-1},v_{j+1},u_{i-1},u_{j+1}\}$ then the proof of this theorem is same as theorem 3.2.

Conclusion

In this paper we presented the number of common vertices in two maximum length paths in 2-*connected* and 3-*connected* graph. It would be interesting to see whether the ideas presented in this paper can be applied for other classes of graphs. It is still open question whether any three maximum length paths of every 2-*connected* and 3-*connected* graph share a common vertex.

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