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Continuous fertility Model and its Bayesian Analysis

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Abstract

The present paper aims at exploring a probability model of continuous fertility and also studies its Bayesian analysis under the linex loss function.

Keyword: Probability model, waiting time, Conception, Linex loss function.

Introduction

The study of women fecundability has controversially been adopted by different workers. The utility of the study depends upon the proper adaption of the fecundability of women. In most of the available literature, it is found that the fecaundability is assumed to be constant for all women, Singh[1]. But in real life there are ample evidences that women vary in their fecundability. So, the fecundability may be thought of as a random variable Henry[2]. The present work deals with the same concept. Let us suppose that fecundability, say θ, follows distribution with p.d.f. g (θ).

Bayesian forecasting is a natural product of a Bayesian approach to inference. TheBayesian approach in general requires explicit formulation of a model, and conditioning on known quantities, in order to draw inferences about unknown ones.

A Bayesian approach might be useful in addressing these issues. By design, Bayesian methods natively consider the uncertainty associated with the parameters of a probability model (even if those uncertain parameters are believed to be fixed numbers). Bayesian methods are often recommended as the proper way to make formal use of subjective information such as expert opinion and personal judgments or beliefs of an analyst. An important advantage of Bayesian methods, unlike frequentist methods with which they are often contrasted, is that they can always yield a precise answer, even when no data at all are available. Finally, recent Bayesian literature has focused on the potential significance of model uncertainty and how it can be incorporated into quantitative analyses.

If T is waiting time of first conception. It can be treated as random variable which follows the distribution with p.d.f. f(x/θ) is regarded as a conditional p.d.f. of X for given θ where marginal probability density function of θ is given by g(θ) the study can be continued.

The Continuous Fertility Model

The geometric distribution is being considered as a discrete model for the waiting time of first conception as developed by Gini [3]the continuous model for the analysis of waiting time of first conception. The intuitive properties of exponential distribution also helped in such considerations. For such analysis Geometric distribution was replaced by the exponential distribution. Thus if x denotes the time of first conception, then its probability density function, say f(x;θ) is given by

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}; \quad x > 0, \quad \theta > 0 \qquad (2.1.1)$$

Where θ is instantaneous fecundability.
The survival function, say S (x) is given by

$$S(x) = P[X > x]$$
$$= \int_x^\infty \frac{1}{\theta} e^{-x/\theta} dx$$
$$= \frac{1}{\theta}\left[\frac{e^{-x/\theta}}{-\frac{1}{\theta}}\right]_x^\infty$$

Or $$S(x)=e^{-x/\theta} \qquad (2.1.2)$$

And the conception rate, say w(x) will be

$$W(x) = \frac{f(x)}{S(x)}$$
$$= \frac{\frac{1}{\theta}e^{-x/\theta}}{e^{-x/\theta}}$$
$$= \frac{1}{\theta} \qquad (2.1.3)$$

Maximum Likelihood Estimator

$$F(\underline{x}/\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$
$$= \left(\frac{1}{\theta}\right)^n e^{-z/\theta} \qquad (2.1.4)$$

Where

$$z = \sum_{i=1}^{n} x_i$$

Bayesian Analysis of the Model

The first conception of the family is also a part of the past family back ground; therefore Bayesian analysis of conception seems realistic on the basis of some history. In some coming section the Bayesian analysis has been done for a continuous time model i.e. exponential distribution.

We have $f(x/\theta) = \theta\, e^{-\theta x};\ x > 0, \theta > 0$

Where θ is the instantaneous fecundibility.
The fundamental problems in Bayesian Analysis are that of the choice of prior distribution g(θ) and a loss function $L(\hat\theta, \theta)$. Let us consider three prior distribution of θ to obtain the Bayes estimators which are as follows:

Quasi-Prior: For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d};\ \theta > 0, d > 0 \qquad (2.1.5)$$

Here d = 0 leads to a diffuse prior and d = 1, a non informative prior.

Natural Conjugate Prior of θ: The most widely used prior distribution of θ is the inverted gamma distribution with parameters α and β (>0) with p.d.f. given by

$$g_2(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}\, \theta^{-(\alpha+1)}\quad e^{-\beta/\theta} & ;\theta > 0\ \ (\alpha,\beta) > 0 \\ 0 & ;\ \text{otherwise} \end{cases} \qquad (2.1.6)$$

The main reason for general acceptability is the mathematical tractability resulting from the fact that inverted gamma distribution is conjugate prior for θ.

Uniform Prior: It frequently happens that the life tester knows in advance that the probable values of θ lies over a finite range [α, β] but he does not have any strong opinion about any subset of values over this range. In such a case uniform distribution over [α, β] may be a good approximation.

$$g_3(\theta) = \begin{cases} \frac{1}{\beta-\alpha}; & 0 < \alpha < \theta \le \beta \\ 0\ ; & \text{otherwise} \end{cases} \qquad (2.1.7)$$

Loss Function: The Bayes estimator $\hat\theta$ of θ is of course, optimal relative to the loss function chosen. A commonly used loss function is the squared error loss function (SELF)

$$L(\hat\theta,\theta) = (\hat\theta - \theta)^2, \qquad (2.1.8)$$

which is a symmetrical loss function and assigns equal losses to over estimation and underestimation. Canfield[4] points out that the use of symmetric loss function may be inappropriate in the estimation of reliability function. Over estimation of reliability function or average lifetime is usually much more serious than under estimation of reliability function or mean failure time. Also, an under estimate of the failure rate results in more serious consequence than an overestimation of the failure rate. This leads to statistician to think about asymmetrical loss functions which have been proposed in statistical literature. It is well known that the Bayes estimator under the above loss function, say $\hat\theta$s, is the posterior mean. The squared error loss function (SELF) is often used also because it does not lead to extensive numerical computation but several authors Ferguson[5], Varian[6], Berger[7], Zellner[8] and Basu and Ebrahimi[9], have recognized the inappropriateness of using symmetric loss function in several estimation problems. These have proposed different asymmetric loss function.

Linex Loss Function: Varian (1975) introduced the following convex loss function known as Linex (Linear – Exponential) loss function.

$$L(\Delta) = b\, e^{a\Delta} - c\Delta - b\ ;\quad a,c \ne 0, b > 0 \qquad (2.1.9)$$

Where $\Delta = \hat\theta - \theta$. it is clear that L (0) = 0 and the minimum occurs when a b = c, therefore, L(Δ) can be written as

$$L(\Delta) = b\,[e^{a\Delta} - a\Delta - 1];\quad a \ne 0, b > 0 \qquad (2.1.10)$$

Where a and b are the parameters of the loss function may be defined as shape and scale respectively. This loss function has been considered by Zellner[8] Rojo[10]. Basu and Ebrahimi[9] who considered the L(Δ)as

$$L(\Delta) = b\,[e^{a\Delta} - a\Delta - 1];\quad a \ne 0, b > 0 \qquad (2.1.11)$$

Where

$$\Delta = \frac{\hat\theta}{\theta} - 1$$

and studied The Bayesian estimation under the asymmetric loss function for exponential life time distribution. This loss function is suitable for the situation where overestimation of θ is more costly than its underestimation.

This loss function L (Δ) have the following nice properties:

(i) For a=1, the function is quite asymmetric about zero with overestimation being more costly than underestimation,
(ii) For a < 0, L(Δ) rises exponentially when Δ< 0 (underestimation) and almost linearly when
Δ> 0 (Overestimation); and
(iii) For small values of |a|

$$L(\Delta) = \frac{b\,a^2\Delta^2}{2} = \frac{ba^2}{\theta^2}\left(\hat\theta - \theta\right)^2$$

is almost symmetric function. Thus, for small values of |a|optimal estimates are not far different form those obtained with a squared error loss function.
Let E_g and E_π denote the prior and posterior expectations, respectively. The posterior expectation of loss function in (2.1.11) is

$$E_\pi[L(\Delta)] = b\left[e^{-a}\, E_\pi\left(\exp\frac{a\,\hat\theta}{\theta}\right) - aE_\pi\left(\frac{\hat\theta}{\theta} - 1\right) - 1\right] \qquad (2.1.12)$$

The value of $\hat\theta$ that minimises (2.1.12), denoted by $\hat\theta_A$, is obtained by solving the following equation

$$\frac{d}{d\hat\theta} E_\pi[L(\Delta)] = 0$$

$$\Rightarrow b\left[ae^{-a}E_\pi\left(\frac{1}{\theta}\exp\frac{a\,\hat\theta}{\theta}\right) - aE_\pi\left(\frac{1}{\theta}\right)\right] = 0$$

Thus Bayes estimator under asymmetric loss L (Δ), i.e., $\hat\theta_A$ is the solution of the following equation

$$E_\pi\left[\frac{1}{\theta}\exp\left(\frac{a\,\hat\theta_A}{\theta}\right)\right] = e^a E_\pi\left(\frac{1}{\theta}\right) \qquad (2.1.13)$$

It may be noted that $E_\pi(\theta)$ is the Bayes estimator under squared error loss function.

Bays Estimator under $g_1(\theta)$: Under $g_1(\theta)$, the posterior distribution is defined by

$$f(\theta|\underline{x}) = \frac{f((\underline{x}|\theta))g_1(\theta)}{\int_0^\infty f((\underline{x}|\theta))g_1(\theta)d\theta} \quad (2.2.1)$$

Substituting the values of $g_1(\theta)$ *and* $f(\underline{x}|\theta)$ from equations (2.1.9) and (2.1.7) in (2.2.1) we get, after simplification, as

$$f(\theta|\underline{x}) = \frac{\left(\frac{1}{\theta}\right)^n\left(\frac{1}{\prod_{i=1}^n x_i}\right)e^{-\left(\frac{z}{\theta}\right)}\frac{1}{\theta^d}}{\int_0^\infty\left(\frac{1}{\theta}\right)^n\left(\frac{1}{\prod_{i=1}^n x_i}\right)e^{-\left(\frac{z}{\theta}\right)}\frac{1}{\theta^d}d\theta} \quad (2.2.2)$$

$$= \frac{z^{n+d-1}}{\Gamma(n+d-1)}\theta^{-(n+d)}e^{-z/\theta} \qquad ;\theta>0, n+d>1.$$

The Bayes estimator under squared error loss function is the posterior mean given by

$$\hat{\theta}_S = \int_0^\infty \theta\, f(\theta|\underline{x})d\theta. \quad (2.2.3)$$

Substituting the values of $f(\theta|\underline{x})$ from equation (2.2.2) in equation (2.2.3) and on solving we get

$$\hat{\theta}_S = \int_0^\infty \frac{z^{n+d-1}}{\Gamma(n+d-1)}\theta^{-(n+d)}e^{-(z/\theta)}d\theta$$

$$= \frac{z^{n+d-1}}{\Gamma(n+d-1)}\int_0^\infty \theta^{-(n+d)}e^{-(z/\theta)}\, d\theta$$

$$= \frac{z^{n+d-1}}{\Gamma(n+d-1)}\frac{\Gamma(n+d-2)}{z^{n+d-2}}$$

$$\hat{\theta}_S = \frac{z}{n+d-2} \quad ; \; n+d>2 . \quad (2.2.4)$$

The Bayes estimator under linex loss function using the value of $f(\theta/\underline{x})$ from equation (2.2.2) is the solution of equation (2.1.13) given by

$$\int_0^\infty \frac{1}{\theta}\left\{exp\left(\frac{a\,\hat{\theta}_A}{\theta}\right)\right\} f(\theta|\underline{x})d\theta = e^a\int_0^\infty \frac{1}{\theta}f(\theta|\underline{x})d\theta$$

On simplification which leads to

$$\hat{\theta}_A = \left(\frac{1-e^{-a/(n+d)}}{a}\right)z. \quad (2.2.5)$$

Bayes Estimator Under $g_2(\theta)$:
Under $g_2(\theta)$, the posterior distribution is defind by

$$f(\theta/\underline{x}) = \frac{f((\underline{x}|\theta))g_2(\theta)}{\int_0^\infty f((\underline{x}|\theta))g_2(\theta)d\theta} \quad (2.3.1)$$

Substituting the values of $g_2(\theta)$ *and* $f(\underline{x}/\theta)$ from equations (2.1.10) and (2.1.7) in (2.3.1) and simplifying, we get

$$f(\theta|\underline{x}) = \frac{\left(\frac{1}{\theta}\right)^n\left(\frac{1}{\prod_{i=1}^n x_i}\right)e^{-\left(\frac{z}{\theta}\right)}\frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\theta}\right)^{\alpha+1}e^{-\beta/\theta}}{\int_0^\infty\left(\frac{1}{\theta}\right)^n\left(\frac{1}{\prod_{i=1}^n x_i}\right)e^{-\left(\frac{z}{\theta}\right)}\frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\theta}\right)^{\alpha+1}e^{-\beta/\theta}d\theta}$$

$$= \frac{(\beta+z)^{n+\alpha}}{\Gamma(n+\alpha)}\theta^{-(n+\alpha+1)}e^{-\frac{1}{\theta}(\beta+z)} \quad (2.3.2)$$

The Bayes estimator under squared error loss function is the posterior mean given by

$$\hat{\theta}_s = \int_0^\infty \theta\, f(\theta|\underline{x})d\theta \quad (2.3.3)$$

Substituting the values of $f(\theta/\underline{x})$ from equation (2.3.2) in equation (2.3.3) and on solving, we get

$$\hat{\theta}_s = \frac{(\beta+z)^{n+\alpha}}{\Gamma(n+\alpha)}\int_0^\infty \theta^{-(n+\alpha)}e^{-\frac{1}{\theta}(\beta+z)}\, d\theta$$

$$= \frac{(\beta+z)^{n+\alpha}}{\Gamma(n+\alpha)}\frac{\Gamma(n+\alpha-1)}{(\beta+z)^{n+\alpha-1}}$$

$$\hat{\theta}_s = \frac{\beta+z}{(n+\alpha-1)} \quad (2.3.4)$$

The Bayes estimator under linex loss function L (Δ), using the value of $f(\theta/\underline{x})$ from the equation (2.3.2) is the solution of equation (2.1.13) given by

$$\int_0^\infty \frac{1}{\theta}\left\{exp\left(\frac{a\hat{\theta}_A}{\theta}\right)\right\} f(\theta|\underline{x})d\theta = e^a\int_0^\theta \frac{1}{\theta}f(\theta|\underline{x})d\theta$$

On simplification which leads to

$$\hat{\theta}_A = \left(\frac{1-e^{-a/(n+\alpha+1)}}{a}\right)(\beta+z) \quad (2.3.5)$$

Bayes Estimator Under $g_3(\theta)$:
Under $g_3(\theta)$, the posterior distribution is defined by

$$f(\theta|\underline{x}) = \frac{f((\underline{x}|\theta))g_3(\theta)}{\int_0^\infty f((\underline{x}|\theta))g_3(\theta)d\theta} \quad (2.4.1.)$$

Substituting the values of $g_3(\theta)$ *and* $f(\theta/\underline{x})$ from equations (2.1.11) and (2.1.7) in (2.4.1) we get, after simplifying, we get

$$f(\theta|\underline{x}) = \frac{\left(\frac{1}{\theta}\right)^n\left(\frac{1}{\prod_{i=1}^n x_i}\right)e^{-\left(\frac{z}{\theta}\right)}\frac{1}{(\beta-\alpha)}}{\int_\alpha^\beta\left(\frac{1}{\theta}\right)^n\left(\frac{1}{\prod_{i=1}^n x_i}\right)e^{-\left(\frac{z}{\theta}\right)}\frac{1}{(\beta-\alpha)}d\theta},$$

$$= \frac{z^{n-1}\theta^{-n}e^{-z/\theta)}}{I_g\left(\frac{z}{\alpha},n-1\right)-\left(\frac{z}{\beta},n-1\right)}, \quad (2.4.2)$$

Where: $I_g(x,n) = \int_0^x e^{-x}x^{n-1}dt$ is the incomplete gama function

The Bayes estimator under squared error loss function is the posterior mean given by

$$\hat{\theta}_S = \int_\alpha^\beta \theta\, f(\theta|\underline{x})d\theta \quad (2.4.3)$$

Substituting the values of $f(\theta/\underline{x})$ from equation (2.4.2) in equation (2.4.3), we get

$$\hat{\theta}_S = \int_\alpha^\beta \theta\frac{z^{n-1}\theta^{-n}e^{-z/\theta}}{I_g\left(\frac{z}{\alpha},n-1\right)-I_g\left(\frac{z}{\beta},n-1\right)}d\theta$$

which on simplification leads to

$$\hat{\theta}_S = \left(\frac{I_g\left(\frac{z}{\alpha},n-2\right)-I_g\left(\frac{z}{\beta},n-2\right)}{I_g\left(\frac{z}{\alpha},n-1\right)-I_g\left(\frac{z}{\beta},n-1\right)}\right)z. \quad (2.4.4)$$

The Bayes estimator under linex loss function, by using the value of from $f(\theta|\underline{x})$ equation (2.4.2) is the solution of equation (2.1.13) given by

$$\int_{\alpha}^{\beta} \frac{1}{\theta}\left\{exp\left(\frac{a\hat{\theta}_A}{\theta}\right)\right\} \mathrm{f}\left(\theta\middle|\underline{x}\right)d\theta = e^{a}\int_{\alpha}^{\beta} \frac{1}{\theta}\mathrm{f}\left(\theta\middle|\underline{x}\right)d\theta$$

On simplification which leads to

$$e^{a}\,\frac{I_g\left(\frac{z}{\alpha},n\right)-I_g\left(\frac{z}{\beta},n\right)}{I_g\left(\frac{z-a\hat{\theta}_A}{\alpha},n\right)-I_g\left(\frac{z-a\hat{\theta}_A}{\beta},n\right)} = \left(\frac{z}{z-a\hat{\theta}_A}\right)^{n} \qquad (2.4.5)$$

The equations (2.4.4) and (2.4.5) can be solved numerically.

Conclusion

Presented Probability model of waiting time of first conception analysed in the Bayesian environment under the precautionary loss function which gives the new dimension of research work in this field it is worth mentioning that for future work. It can be expend this type of work under the different situation in the Bayesian environment in the field of demography.

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