Research Journal of Mathematical and Statistical Sciences ______________________________ ISSN 2320-6047
Vol. **11(1),** 1-3, May **(2023)** *Res. J. Mathematical and Statistical Sci.*

On Solutions of nonlinear functional integral equations in Frechet space

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Available online at:www.isca.in, www.isca.me
Received 6th December 2021, revised 13th October 2022, accepted 8th February 2023

Abstract

In this article, a special type of integral equation will be investigated. The interest in this study to show existence of a unique solution on a semi-infinite interval in Frechet space by using a nonlinear alternative of the Leray-Schauder type for contraction maps.

Keywords: Functional integral equation, existence and uniqueness, fixed point, Leray-Schauder, Frechet space.

Introduction

In mathematics, integral equations are one of the most important topics. It has a wide range of applications in a variety of fields from physics, engineering, population dynamics and economics[1-5]. In order to solve integral equations, researchers have devised a various techniques. One of the used and effective tool is Fixed-point theorem, it has been a powerful method for demonstrating the existence and uniqueness of solutions to a broad types of nonlinear integral equations[6-9].

In this paper, Leray-Schauder Alternatives[10], has been used to prove that is an exist of a unique solution for nonlinear functional integral equations

$$k(t) = U(t,s) + A\big(k(t)\big)$$
$$\left(\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f\big(t,s,k(s)\big)ds \int_0^T \sum_{i=1}^{k} C_i(t,s)\, v(t,s,k(s))ds\right), t \in I = [0,+\infty) \quad (1),$$

which defined on a semi-finite interval $I = [0,+\infty)$, where $U\colon \mathrm{I} \times \mathrm{I} \to \mathbb{R}, C_i\colon \mathrm{I} \times I \to \mathbb{R}, D_i\colon \mathrm{I} \times [0,T] \to \mathbb{R}, f\colon \mathrm{I} \times [0,T] \times \mathbb{R} \to \mathbb{R}$, $v\colon \mathrm{I} \times [0,T] \times \mathbb{R} \to \mathbb{R}$are maps and $A\colon C(I,\mathbb{R}) \to C(I,\mathbb{R})$ is an appropriate operator, $C(I,\mathbb{R})$ indicates the continuing function space $k\colon I \to \mathbb{R}$.

The methodology is inspired and motivated by Enchohra and Darwish[11], the authors studied the existence of a unique solution for Urysohn type

$$k(t) = f(t) + (Ak)(t)\int_0^T u(t,s,k(s))ds, \quad t \in [0,+\infty) (2)$$

Also, in Sadoon[12], authors showed an investigation into the possibility of finding a solution that is unique to the nonlinear quadratic integral equation of the Fredholm-Volterra integral equation

$$k(t) = f(t) + (Ak)(t)$$
$$\left[\int_0^T u\big(t,s,k(s)\big) + \int_0^t g\big(t,s,k(s)\big)\right]ds, \quad t \in [0,+\infty) \qquad (3)$$

Equation (1) is more general and comprehensive than the Equation (2) and Equation (3) two, moreover, the necessary conditions for achieving the desired result have been established.

The remainder of the article is planned as follows: Hypotheses that will be used in the later section are presented in Section 2; the main result is presented in Section 3.

Methodology

To begin, this section introduces some formulas, definitions, and theorems that will be used throughout the rest of this study.

Let $\mathcal{B} \subset \mathcal{M}$, where $\mathcal{M}$ imposed to be Frechet space with a family of semi-norms $\{\|\|_n\}_{n\in N}$, therefor $\mathcal{B}$ is bounded if for each $\mathrm{n} \in \mathbb{N}$, there exists $\mho_\mathrm{n} > 0$ such that

$\| \sigma \|_\mathrm{n} \leq \mho_\mathrm{n}$ for all $\sigma \in \mathcal{B}$

Theorem 1 [11]: Let Ψbe a closed subset of a Frechet space X such that $0 \in \Psi$ and $\digamma\colon \Psi \to X$ is a contraction such that $\digamma(\Psi)$ is bounded. Then either- i. $\digamma$ has a unique fixed point or ii. There exists $\alpha \in (0,1), \mathrm{n} \in \mathrm{N}$ and $\beta \in \partial\Psi^\mathrm{n}$ such that $\| \beta - \alpha\digamma(\beta) \|_\mathrm{n} = 0$.

Where $\partial\Psi^\mathrm{n}$ is boundary of Ψ^n.

Results and discussion

In this section, we assume that the following assumptions are satisfied: i. $U\colon \mathrm{I} \times I \to R$ is a continuous function, with B_n=sup$|\mathrm{U(t,s)}|$, ii. For all $n \in N \exists$ $\cup_z^* > 0$ s.t. $|(Ak)(t) - (Ay)(t)| \leq \cup_z^* |k(t) - \mathrm{y}(t)|$ for all $k, y \in C(I,\mathbb{R})$ and $t \in$

$[0,n]$. iii. ∃ two positive constants a, b ∋: $|(Ak)(t)| \leq a + b|k(t)|$ for each $k \in C(I,\mathbb{R})$ and $t \in I$. iv.D_i, C_i: I × $I \to \mathbb{R}$ is continuing bounded maps therefore, there exist a positive constant
$Z_1^*, Z_2^* > 0 \ni \sum_{i=1}^{k} |D_i(t,s)| \leq Z_1^*, and \sum_{i=1}^{k} |C_i(t,g)| \leq Z_2^*$, where (t,s),(t,g) $\in I$, v. f: I × $[0,T] \times \mathbb{R} \to \mathbb{R}$ is continuing map and for each $n \in \mathbb{N}$, ∃ a positive constant $L_h^* > 0$∋:

$|f(t,s,k) - f(t,s,y)| \leq L_h^* |k-y|$ for all $(t,s) \in [0,T]$ and $k, y \in \mathbb{R}$

∃ a continuing nondecreasing map $\psi: I \to (0,\infty)$ and $p \in C(I,\mathbb{R}_+) \ni |f(t,s,k)| \leq p(s)\vartheta(|k|)$ for all $(t,s) \in I \times [0,T], k \in \mathbb{R}$

v: I × $[0,T] \times \mathbb{R} \to \mathbb{R}$ is continuing map and for each $n \in \mathbb{N}$, ∃ a positive constant $H_n^* > 0$∋:
$|v(t,s,k) - v(t,s,y)| \leq H_n^* |k-y|$ for all $(t,s) \in [0,T]$ and $k, y \in \mathbb{R}$

∃ a continuing nondecreasing map δ: I $\to (0,\infty)$ with:
$q \in C(I,\mathbb{R}_+)$Such that $|v(t,s,k)| \leq q(s)\delta(|k|)$ for each $(t,s) \in I \times [0,T], k \in \mathbb{R}$, and ∃ constants $\mho_n$, n ∈ ℕ such that:

$$\frac{M_n}{B_n + (a + b(\|\varphi\|_n) T^2 Z_1^* Z_2^* p^* q^* (\vartheta(\|\varphi\|_n)\delta(\|\varphi\|_n))} > 1, \qquad (4)$$

Where: $p^* = \sup\{p(s): s \in [0,T]\}$, $q^* = \sup\{q(s): s \in [0,T]\}$.

Theorem 2: Suppose that hypotheses (a-i) are satisfied. If

$T Z_1^* \{(a + bM_n)[Z_2^* p^*(s)\vartheta(M_n) H_n^* + L_h^*] + Z_2^* q^*(s)\delta(M_n) p^*(s)\vartheta(M_n) \cup_z^*\} < 1(5)$

Then Equation (1) has a unique solution.

Proof: Let C(I, ℝ) be the semi-norms by $\| y \|_n = \sup|y(t)|, t \in [0,n]$, for each n ∈ ℕ, then $C(I,\mathbb{R})$ is a Frechet space with the family of semi-norms $\{\|\|\}_{n\in N}$,[11].

Convert the Equation (1) into a fixed point equation. With taking consideration the operator

F: C(I, ℝ) → C(I, ℝ) Introduced by

$$(F\varphi)(t) = U(t,s) + A(\varphi(t)) \left(\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\varphi(s)) ds \int_0^T \sum_{i=1}^{k} C_i(t,g)\, v(t,s,\varphi(s)) dg\right), t \in I$$

By assuming that φ is a solution of the Equation (1). n ∈ ℕ is given, and t ≤ n, therefore from (i), (d), (f) and (h) we have:

$$|\varphi(t)| \leq |U(t,s)| + |A(\varphi(t))|$$

$$\left(\int_0^T \sum_{i=1}^{k} |D_i(t,s)|\, |f(t,s,\varphi(s))| ds \int_0^T \sum_{i=1}^{k} |C_i(t,g)|\, |v(t,s,\varphi(s))| dg\right)$$

$\leq B_n + (a + b|\varphi(t)|)$
$\left(\int_0^T Z_1^* p(s)\vartheta(|\varphi(s)|) ds \int_0^T Z_2^* q(s)\delta(|\varphi(s)|) ds\right)$,

$\| \varphi \|_n \leq B_n + (a + b(\| \varphi \|_n) T^2 Z_1^* Z_2^* p^* q^* (\vartheta(\| \varphi \|_n)\delta(\| \varphi \|_n))$,

then

$$\frac{\| \varphi \|_n}{B_n + (a + b(\| \varphi \|_n) T^2 Z_1^* Z_2^* p^* q^* (\vartheta(\| \varphi \|_n)\delta(\| \varphi \|_n))} \leq 1$$

From (3) for all $n \in N$, $\| \varphi \|_n \neq \mho_n$ suppose,

$\Omega = \{\varphi \in C(I,R): \| \varphi \|_n \leq M_n$ for all $n \in N\}$

It is obvious, Ω is a closed set subset of $C(I,\mathbb{R})$, we need to demonstration that $F: \Omega \to C(I,R)$ is a contraction mapping, let $\varphi, \psi \in \Omega$ for each $t \in [0,n]$ and $n \in \mathbb{N}$ from (b)-(i) we get:

Let $\varphi, \psi \in \Omega$, then $F\varphi(t) - F\psi(t) = A(\varphi(t))$

$$\left(\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\varphi(s)) ds \int_0^T \sum_{i=1}^{k} C_i(t,g)\, v(t,g,\varphi(g)) dg\right)$$

$-A(\psi(t)) \left(\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\psi(s)) ds \int_0^T \sum_{i=1}^{k} C_i(t,g)\, v(t,g,\psi(g)) dg\right)$

By adding and subtracting
$A(\varphi(t)) \left(\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\varphi(s)) ds \int_0^T \sum_{i=1}^{k} C_i(t,g)\, v(t,g,\psi(g)) dg\right)$,
we get =

$A(\varphi(t)) \left(\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\varphi(s)) ds \int_0^T \sum_{i=1}^{k} C_i(t,g)\, v(t,g,\varphi(g)) dg\right)$

$-A(\psi(t)) \left(\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\psi(s)) ds \int_0^T \sum_{i=1}^{k} C_i(t,g)\, v(t,g,\psi(g)) dg\right)$

$+A(\varphi(t)) \left(\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\varphi(s)) ds \int_0^T \sum_{i=1}^{k} C_i(t,g)\, v(t,g,\psi(g)) dg\right)$

$- A(\varphi(t)) \left(\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\varphi(s)) ds \int_0^T \sum_{i=1}^{k} C_i(t,g)\, v(t,g,\psi(g)) dg\right)$

Furthermore, by adding and subtracting
$A(\psi(t)) \int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\varphi(s)) ds$, we get =

$A(\varphi(t))[\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\varphi(s)) ds \int_0^T \sum_{i=1}^{k} C_i(t,g)\, [v(t,g,\varphi(g)) - v(t,g,\psi(g))] dg]$
$+\int_0^T \sum_{i=1}^{k} C_i(t,g)\, v(t,g,\psi(g)) dg [\int_0^T \sum_{i=1}^{k} D_i(t,s)\, f(t,s,\varphi(s)) ds \{A(\varphi(t)) - A(\psi(t))\} + A(\psi(t)) \{\int_0^T \sum_{i=1}^{k} D_i(t,s)\, [f(t,s,\varphi(s)) - f(t,s,\psi(s))]\} ds]$

Therefore, $|F\varphi(t) - F\psi(t)| =$

$$|A(\varphi(t))|[\int_0^T \sum_{i=1}^{k} D_i(t,\mathrm{s})\, \mathrm{f}(\mathrm{t},\mathrm{s},\varphi(\mathrm{s}))\mathrm{ds} \int_0^T \sum_{i=1}^{k} C_i(t,\mathrm{g})\, [\mathrm{v}(\mathrm{t},\mathrm{g},\varphi(\mathrm{g})) - \mathrm{v}(\mathrm{t},\mathrm{g},\psi(\mathrm{g}))]\mathrm{dg}]$$
$+$
$\int_0^T \sum_{i=1}^{k} C_i(t,g)\, \mathrm{v}(\mathrm{t},\mathrm{g},\psi(\mathrm{g}))\mathrm{dg}\left[\{\int_0^T \sum_{i=1}^{k} D_i(t,\mathrm{s})\, \mathrm{f}(\mathrm{t},\mathrm{s},\varphi(\mathrm{s}))\mathrm{ds}\{A(\varphi(t)) - A(\psi(t))\} + +A(\psi(t))\{\int_0^T \sum_{i=1}^{k} D_i(t,\mathrm{s})\, [\mathrm{f}(\mathrm{t},\mathrm{s},\varphi(\mathrm{s})) - \mathrm{f}(\mathrm{t},\mathrm{s},\psi(\mathrm{s}))]\right\}\mathrm{ds}|$

Hence,

$$|F\varphi(t) - F\psi(t)| \leq$$
$$|A(\varphi(t))|[\int_0^T \sum_{i=1}^{k} |D_i(t,\mathrm{s})||\, \mathrm{f}(\mathrm{t},\mathrm{s},\varphi(\mathrm{s}))|\mathrm{ds} \int_0^T \sum_{i=1}^{k} |C_i(t,\mathrm{g})\, |[|\mathrm{v}(\mathrm{t},\mathrm{g},\varphi(\mathrm{g})) - \mathrm{v}(\mathrm{t},\mathrm{g},\psi(\mathrm{g}))|]\mathrm{dg}]$$

$+$
$\int_0^T \sum_{i=1}^{k} |C_i(t,g)||\, \mathrm{v}(\mathrm{t},\mathrm{g},\psi(\mathrm{g}))|\mathrm{dg}\left[\{\int_0^T \sum_{i=1}^{k} |D_i(t,s)||\, \mathrm{f}(\mathrm{t},\mathrm{s},\varphi(\mathrm{s}))|\mathrm{ds}\{|A(\varphi(t)) - A(\psi(t))|\} + |A(\psi(t))|\{\int_0^T \sum_{i=1}^{k} |D_i(t,\mathrm{s})\, |[\mathrm{f}(\mathrm{t},\mathrm{s},\varphi(\mathrm{s})) - \mathrm{f}(\mathrm{t},\mathrm{s},\psi(\mathrm{s}))]|\right\}\mathrm{ds}$

$\leq a + b|\varphi(t)||[\int_0^T Z_1^* \mathrm{p(s)}\vartheta(|$
$\mathrm{x}\, |)\mathrm{ds}\int_0^T Z_2^* H_n^* |x-y|\mathrm{dg}] + \int_0^T Z_2^* \mathrm{q(s)}\delta(|\,\psi\,|)\mathrm{d}g$
$\times \int_0^T Z_1^* \mathrm{p(s)}\vartheta(|\,\mathrm{x}\,|) \cup_z^* |x-y|ds + a + b|\varphi(t)|[\int_0^T Z_1^* L_h^* |x -$
$|ds]$
$\leq a + bM_n[TZ_1^* \mathrm{p}^*(\mathrm{s})\vartheta(M_n) TZ_2^* H_n^* |x-y|]$
$+ TZ_2^* \mathrm{q}^*(\mathrm{s})\delta(M_n)\mathrm{T}Z_1^* \mathrm{p}^*(\mathrm{s})\vartheta(M_n) \cup_z^* |x-y|$
$+a + bM_n[TZ_1^* L_h^* |x-y|]$
$\leq TZ_1^* \{(a + bM_n)[Z_2^* \mathrm{p}^*(\mathrm{s})\vartheta(M_n) H_n^* + L_h^*]$
$+ Z_2^* \mathrm{q}^*(\mathrm{s})\delta(M_n)\mathrm{p}^*(\mathrm{s})\vartheta(M_n) \cup_z^*\}|x-y|$

According to (5) F is a contraction mapping for all $n \in \mathbb{N}$. Based on the selection of Ω is no $\varphi \in \partial\Omega$ such that $\varphi = \lambda F(\varphi)$ for certain $\lambda \in (0,1)$. Therefor, the requirement 2* in the theorem (1) does not satisfy. As result theorem (2) demonstrates that requirement 1* satisfy, and then the map F has a solution for Eq (1), and it is a unique fixed point y in Ω .

Conclusion

The works based on a nonlinear alternative of the Leray-Schauder type for contraction theory, where this theory guarantee that a solution exist for Equation (1) which is unique, in order to achieve theory conditions, we provide appropriate conditions placed on Section 3. In addition, Equation (1) was more general from Equation (2) and Equation (3). This result sets the stage for further investigation into the stability of Equation (1).

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