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Common Fixed Point Theorem for four Mappings in Fuzzy Metric Space

Namdeo Kailash[1], Shrivastav Rajesh[2], Solanki Manoj[3] and Sharma Manoj[4]
[1]Deptt. of Mathematics, Govt. PG College Pipariya, MP, INDIA
[2]Deptt. of Mathematics, Govt. Science and Commerce College Benazir, Bhopal, MP, INDIA
[3]Deptt. of Mathematics, Sadhuvashwani College Bairagarh, Bhopal, MP, INDIA
[4]Deptt. of Mathematics, Sagar Institute of Research and Technology, Bhopal, MP, INDIA

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Abstract

In the Present research paper, we prove common fixed point theorem for four mapping using new condition in fuzzy metric spaces.

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Introduction

The concept of a fuzzy set was first introduced by Zadeh L.A.[1] and developed abasic frame work to treat mathematically the fuzzy phenomena or systems which due to in trinsic indefiniteness, cannot themselves be characterized precisely. fuzzy metric spaces have been introduced by Kramosil and Michalek[2] and George and Veersamani[3] modified the notion of fuzzy metric with help of continuous t-norms. Recently many have proved fixed point theorems involving fuzzy sets[4-12,18-21] Balasubramaniam P., Muralishankar S.R. and Pant R.P.[4] proved the open problem of Rhoades[17] on the existence of a contractive definition which generals a fixed point but does not force the mapping to be continuous at the fixed point possesses an affirmative answer.

Definition-1: The 3-tuple (X, S, *) is said to be a S-Fuzzy Metric Space if X is an arbitrary Set, * is a continuous t-norm and S is a Fuzzy set on $X^2 \times (0, \infty)$. satisfying the following conditions: i. $S(x, y, t) > 0$, ii. $S(x, y, t) = 1$ if and only if $x=y$, iii. $S(x, y, t)= S(y, x, t)$, iv. $S(x, y, t) * S(y, z, s) \leq S(x, z, t+s)$, v. $S(x, y, .);\ (0, \infty) \to [0, 1]$ is continuous for all $x, y, z \in X$ and $t, s > 0$.

Definition-2: A sequence $\{x_n\}$ in a Fuzzy Metric Space (X, S, *) is a Cauchy Sequence if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $S(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$

Definition-3: A sequence $\{x_n\}$ in a Fuzzy Metric Space (X, S, *) is converges to x if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $S(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

Definition-4: Fuzzy Metric Space (X, S, *) is said to be complete if every Cauchy Sequence in (X, S, *) is a convergent sequence.

Definition-5: Two mappings f and g of a fuzzy metric space (X, S, *) in to itself are said to be weakly commuting if $S(fgx, gfx, t) \geq S(fx, gx, t)$ for each x in X.

Definition-6: The mappings f and g of a fuzzy metric space (X, S, *) in to itself are said to be R-weakly commuting, provided there exists some positive real numbers R such that $S(fgx, gfx, t) \geq S(fx, gx, t/R)$ for each x in X.

Definition-7: The mappings F and G of a fuzzy metric space (X, S, *) in to itself are said to be compatible iff $S(FGx_n, GFx_n, t) \to 1$ For all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Fx_n, Gx_n \to y$ for some y in X.

Definition-8: Let A and B be self mappings of a fuzzy metric space (X, S, *) ,we will call A and B to be reciprocally continuous if $\lim_{n\to\infty} ABx_n = Ap$ and $\lim_{n\to\infty} BAx_n = Bp$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n= \lim_{n\to\infty} Bx_n=p$ for some p in X

If A and B are continuous then they are obviously reciprocally continuous. But the converse need not be true.

Theorem-1: Let A, B, M and N be self maps of a complete fuzzy metric space (X, S, *) with continuous t – norm * defined by a*b = min {a,b}, a,b $\in [0,1]$ satisfying the following conditions: i. $A(x) \subset N(x)$, $B(x) \subset M(x)$, ii. [A, M], [B, N] are pointwise R-weakly commuting pairs of maps. iii. [A, M] or [B, N] is compatible pair of reciprocally continuous maps. iv. For all x, y in X, $k \in [0,1]$ t>0, $S^2(Ax, By, kt) \geq \max\{ S^2(Mx, Ny, t), S^2(Ax, Mx, t), S^2(By, Ny, t) \}$, v. For all x, y in X, $\lim S(x, y, t) \to 1$ as $t \to \infty$

Then A, B, M and N have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Construct a sequence $\{y_n\}$ such that
$y_{2n-1} = Nx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Mx_{2n} = Bx_{2n-1}$ n=1,2,3,……….

Now using (iv) we have

$$\begin{aligned} S^2(y_{2n+1}, y_{2n+2}, kt) &= S^2(Ax_{2n}, Bx_{2n+1}, kt) \\ &\geq \max\{ S^2(Mx_{2n}, Nx_{2n+1}, t), S^2(Ax_{2n}, Mx_{2n}, t), S^2(Bx_{2n+1}, Nx_{2n+1}, t) \} \\ &\geq \max\{ S^2(y_{2n}, y_{2n+1}, t), S^2(y_{2n+1}, y_{2n}, t), S^2(y_{2n+2}, y_{2n+1}, t) \} \\ &\geq \max\{ S^2(y_{2n}, y_{2n+1}, t), S^2(y_{2n+1}, y_{2n+2}, t) \} \end{aligned}$$

$$\Rightarrow S(y_{2n+1}, y_{2n+2}, kt) \geq S(y_{2n}, y_{2n+1}, t) \qquad (1.1)$$

Further using (iv) we have

$$\begin{aligned} S^2(y_{2n}, y_{2n+1}, kt) &= S^2(Bx_{2n-1}, Ax_{2n}, kt) \\ &= S^2(Ax_{2n}, Bx_{2n-1}, kt) \\ &\geq \max\{ S^2(Mx_{2n}, Nx_{2n-1}, t), S^2(Ax_{2n}, Mx_{2n}, t), S^2(Bx_{2n-1}, Nx_{2n-1}, t), \} \\ &\geq \max\{ S^2(y_{2n}, y_{2n-1}, t), S^2(y_{2n+1}, y_{2n}, t), S^2(y_{2n}, y_{2n-1}, t) \} \end{aligned}$$

$$\Rightarrow S(y_{2n}, y_{2n+1}, kt) \geq S(y_{2n-1}, y_{2n}, t) \qquad (1.2)$$

Using (1.1) and (1.2) we have

$$\begin{aligned} S(y_n, y_{n-1}, (1-k)t/k) &\geq S(y_{n-1}, y_{n-2}, (1-k)t/k^2) \\ &\geq S(y_{n-2}, y_{n-3}, (1-k)t/k^3) \\ &\text{----------------------------} \\ &\text{----------------------------} \\ &\text{----------------------------} \\ &\text{----------------------------} \\ &\geq S(y_0, y_1, (1-k)t/k^n) \to 1 \text{ as } n \to \infty \end{aligned}$$

Hence for t > 0, k, $\lambda \in (0, 1)$ we can choose $n_0 \in N$ such that

$$S(y_n, y_{n-1}, (1-k)t/k) \geq 1- \lambda \qquad n \geq n_0 \qquad (1.3)$$

To prove that $\{y_n\}$ is a Cauchy Sequence, we claim (3.1.4) is true for all $n \geq n_0$ and for every $m \in N$

$$S(y_n, y_{n+m}, t) \geq 1- \lambda \qquad (1.4)$$

From (1.1) (1.2) and (1.3) we have

$$\begin{aligned} S(y_n, y_{n+1}, t) &\geq S(y_n, y_{n-1}, t/k) \\ &\geq S(y_n, y_{n-1}, (1-k)t/k) \\ &\geq 1- \lambda \end{aligned}$$

Thus result (1.4) is true for m = 1. Further suppose (1.4) is true for m.

Then we shall show that it is also true for m+1.

Using (1.1) (1.2) and definition for t – norm we have

$$\begin{aligned} S(y_n, y_{n+m+1}, t) &\geq S(y_{n-1}, y_{n+m}, t/k), \\ &\geq \min(S(y_n, y_{n-1}, (1-k)t/k), S(y_n, y_{n+m}, t)\} \\ &\geq 1- \lambda \end{aligned}$$

Thus (1.4) is true for m+1 and so it is true for every m $\in$ N therefore $\{y_n\}$ is a Cauchy Sequence.
Since (X, S, *) is complete so $\{y_n\}$ converges to some point z in X. Thus $\{Ax_{2n}\}$ $\{Mx_{2n}\}\{Bx_{2n-1}\}$ and $\{Nx_{2n-1}\}$ also converges to z. Suppose [A, M] is a compatible pair of reciprocally continuous maps. Then by the definition of reciprocally continuous maps $AMx_{2n} \rightarrow Az$ and $MAx_{2n} \rightarrow Mz$ And then the compatibility of A and M yields $\lim_{n\to\infty} S(AMx_{2n}, MAx_{2n}, t) = 1$

i.e. S(Az, Mz, t) = 1

Hence Az = Mz Since A(x) $\subset$ N(x), There exists a point w in X such that Az = Nw

Using (iv) we have

$S^2(Az, Bw, kt) \geq \max \{ S^2(Mz, Nw, t), S^2(Az, Mz, t), S^2(Bw, Nw, t)\}$
$\geq \max \{ S^2(Az, Az, t), 1, S^2(Bw, Az, t) \}$

Or

$S^2(Az, Bw, kt) \geq 1$

Which implies that Az=Bw, thus Mz = Az = Nw = Bw.

Point-wise R-weakly commutativity of A and M implies that there exists R > 0 such that $S(AMz, MAz, t) \geq S(Az, Mz, t/R) = 1$
i.e. AMz = MAz and AAz = AMz = MAz = MMz

Similarly pointwise R-weakly commutativity of B and N implies that
BBw = BNw = NBw = NNw

Now by (iv) we have

$S^2(AAz, Az, kt) = S^2(AAz, Bw, kt)$
$\geq \max \{ S^2(MAz, Nw, t), S^2(AAz, MAz, t), S^2(Bw, Nw, t)\}$
$\geq \max \{ S^2(AAz, Az, t), 1, S^2(Az, Az, t) \}$

Or

$S^2(AAz, Az, kt) \geq 1$

$\Rightarrow$ AAz = Az thus Az = AAz = MAz

Thus Az is a common fixed point of A and M

Again by (iv) we have

$S^2(Az, BBw, kt) \geq \max \{ S^2(Mz, NBw, t), S^2(Az, Mz, t), S^2(BBw, NBw, t)\}$
$\geq \max \{ S^2(Az, BBw, t), 1, S(BBw, Az, t), \}$

Or $S^2(Az, BBw, kt) \geq 1$

$\Rightarrow$ Az = BBw thus Az = BBw = Bw

Thus Bw(=Az) is a common fixed point of B and N and hence Az is a common fixed point of A, B, M and N.

To prove Uniqueness, let Az_1 be another common fixed point of A, B, M and N. Then we have

$S^2(Az, Az_1, kt) = S^2(AAz, BAz_1, kt)$
$\geq \max \{ S^2(MAz, NAz_1, t), S^2(AAz, MAz, t), S^2(BAz_1, NAz_1, t)\}$
$\geq \max \{ S^2(Az, Az_1, t), S^2(Az, Az, t), S^2(Az_1, Az_1, t) \}$
$\geq \max \{ S^2(Az, Az_1, t), 1, 1\}$

or $S^2(Az, Az_1, kt) \geq 1$

$\Rightarrow Az = Az_1$

Thus Az is a unique common fixed point of A, B, M and N.

Conclusion

Theorem 1 extends the generalize results Balasubramaniam and Muralishankar S., Pant R.P.[4] on the existence of a contractive definition which generals a fixed point but does not force the mapping to be continuous at the fixed point.

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