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# New Types of Separation Axioms VIA Generalized B- Open Sets

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## Abstract

*In this work we introduce and study new types of separation axioms termed by, generalized b- $R_i$, i= 0, 1 and generalized b- $T_i$, i= 0, 1, 2 by using generalized- b open sets due to Ganster and Steiner. Relations among these types are investigated. Several properties and characterizations are provided. Furthermore, a new characterization of $T_{3/4}$ space is obtained. It is also seen that digital line is b- $R_1$, b- $R_0$ and gb- $T_1$. 2000 Math. Subject Classification: 54A05, 54C05, 54D10.*

**Keywords and Phrases:** gb- closed sets, gb- open sets, gb- $R_i$, i= 0, 1, gb- $T_i$, i= 0, 1, 2, $T_{3/4}$, digital line..

## Introduction

In 1996, Andrijevic[1] introduced a class of open sets called b- open sets in topology. In 1970 Levine[2] introduced the concept of generalized closed sets. Ganster and Steiner[3] generalized the concept of closed sets to b- generalized closed sets and generalized b- closed sets. The investigation on generalization of closed set has lead to significant contribution to the theory of separation axioms. Some separation axioms are useful in computer science, as an example, Dontcheve and Ganster[4] proved that the Khalimsky line or the digital line is $T_{3/4}$ space but not $T_1$. Navalagi[5] introduce semi generalized- $T_i$ spaces, i= 0, 1, 2. This paper is devoted to introduce a new class of separation axioms called generalized b- $R_i$ ( briefly gb- $R_i$), i= 0, 1 and generalized b- $T_i$ ( briefly gb- $T_i$), i= 0, 1, 2 axioms using gb- open sets due to Ganster and Steiner[3] . We study basic properties and preservation properties of these spaces. Further, we show that the digital line is b-$T_1$, b- $R_1$, b- $R_0$ and gb- $T_1$.

## Preminilaries

Through this paper X and Y denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be the subset of the space X. The interior and closure of a set $A$ in X are denote by $cl(A)$ and $\text{int}(A)$ respectively. The complement of A is denoted by (X- A) or $A^c$.

Let us recall some definitions and results which are useful in the sequel.

**Definiton 2.1.** Let A be a subset of a space X, then A is called b- open[1] (resp. semi- open[6], regular open[7], preopen[8]) if $A \subseteq cl(\text{int}(A)) \bigcup \text{int}(cl(A))$ (resp. $A \subseteq cl(\text{int}(A))$, $A = \text{int}(cl(A))$, $A \subseteq \text{int}(cl(A))$). The set of all b- open (resp. semi- open) sets is denoted by BO(X) (resp. SO(X)). The complement of the above sets are called their respective closed sets.

**Definition 2.2.** (1) The b- closure (resp. b- interior) of a set A, denoted by bcl(A) (resp. bint(A)) is the intersection (resp. the union) of all b- closed (resp. all b- open) sets containing A( resp. contained in A)[1].

**Definition 2.3**. A subset $A$ of a topological space $(X,\tau)$ is said to be generalized b- open (briefly gb- open) set if $U \subseteq bint\ (A)$ whenever $U \subseteq A$ and $U$ is closed. The complement of generalized- b open set is said to be generalized b- closed. The family of all gb- open (resp. gb- closed) sets of X is denoted by GBO(X) (resp. GBC(X))[3].
It is known that the union (resp. intersection) of two gb- closed sets is not a gb- closed set.

**Definition 2.4.** (1) The gb- closure ( resp. gb- interior ) of A, denoted by gbcl(A) (resp. gbint(A)) is the intersection of all gb- closed (resp. the union of all gb- open) sets containing A( resp. contained in A).

It is easy to see that,

**Remark 2.5.** (1) closedness $\Rightarrow$ b- closedness $\Rightarrow$ gb- closedness. Hence, openess $\Rightarrow$ b- openess $\Rightarrow$ gb- openness. And for any subset A of X, (2) $gbcl(A) \subseteq bcl(A) \subseteq cl(A)$. (3) $int(A) \subseteq bint(A) \subseteq gbint(A)$.

**Remark 2.6.** (1) $A \subseteq B \Rightarrow gbcl(A) \subseteq gbcl(B)$. (2) If *A* is gb- closed, then *gbcl*(*A*)= *A*.
It is easy to prove the following result

**Proposition 2.7.** Let X be a space and $A \subset X$, then $x \in gbcl(\{A\})$ if and only if for each gb- open set U containing x, $A \cap U \neq \phi$.
Al- Omeri et al.[9] presented the definition of gb- irresolute.

**Definition 2.8.** A map *f*: (*X*, $\tau$) $\longrightarrow$ (*Y*, $\sigma$) is said to be gb- irresolute if for each gb- closed set F of *Y*, the inverse image $f^{-1}(F)$ is a *gb*-closed set in X.
Hussein [10] introduced the following definitions

**Definition 2.9.** A map *f*: (*X*, $\tau$) $\longrightarrow$ (*Y*, $\sigma$) is said to be pre- gb- open if for each gb- open set *U* of *X*, the image *f*(*U*) is gb-open set in ***Y***.

**Definition 2.10.** A bijection map *f*: (*X*, $\tau$) $\longrightarrow$ (*Y*, $\sigma$) is said to be gbc- homeomorphism if f is gb- irresolute and pre- gb- open (equivalently, *f* and $f^{1}$ are gb- irresolute) and hence we say that X and Y are gbc- homeomorphic.

**Definition 2.11.** A property p of a topological space X is called a gbc- topological property if every space Y gbc- homeomorphic to X also has the same property.

**Definition 2.12. A** topological space X is called[11],
i. b- $T_0$ if for any two points x, y of X such that x ≠y, there is a b-open set containing one of the two points but not the other.
ii. b- $T_1$ if for any two points x, y of X such that x≠ y, there are two b-open sets, one contains x but not y and the other contains y but not x.
iii. b- $T_2$ if for each pair of distinct points x, y of X, there exists a pair of disjoint b-open sets one contains x and the other contains y.

**Remark 2.13.** In definition 2.12, if we replace each b- open set by semi- open set, we obtain the definitions of semi- $T_0$, semi- $T_1$ and semi- $T_2$ spaces which are given by Maheshwari[12], et al.

**Remark 2.14.** (1) Every topological space is b- $T_0$ (Caldas and Jafari[13]). (2) b-$T_2$ $\Rightarrow$ b-$T_1$ $\Rightarrow$ b- $T_0$ is given by Mustafa.[11] (3) semi- $T_i \Rightarrow$ b- $T_i$ for i= 0, 1, 2.
Mustafa[11] proved that,

**Proposition 2.15.** A space X is b-$T_1$ if and only if every singleton is b- closed.
Dontchev and Ganster[4] proved that,

**Proposition 2.16.** A space X is $T_{3/4}$ if and only if every singleton is regular open or closed.

## gb-$R_0$ Spaces and gb- $R_1$ Spaces

In this section, we define and study two kinds of separation axioms namely, gb- $R_0$ and gb- $R_1$ spaces. Characterizations and properties of these spaces are provided.

**Definition 3.1.** We say that a space X is a gb- $R_0$ space if every gb- open set contains the gb- closure of each of its singletons.

**Definition 3.2.** We say that a space X is a gb- $R_1$ if for any x, y in X with $gbcl(\{x\}) \neq gbcl(\{y\})$, there exist disjoint gb- open sets U and V such that $gbcl(\{x\})$ is a subset of U and $gbcl(\{y\})$ is a subset of V.

**Proposition 3.3.** Every gb- $R_1$ is gb- $R_0$.

**Proof.** Let U be a gb- open set such that $x \in U$. If $y \notin U$, then $x \notin gbcl\{y\}$ and hence $gbcl\{x\} \neq gbcl\{y\}$. Then there is a gb- open set V such that $y \in V$ and $gbcl\{y\} \subset V$ and $x \notin V$, hence $y \notin gbcl\{x\}$. Thus $gbcl\{x\} \subset U$. Thus X is gb- $R_0$.

**Theorem 3.4.** The following statements are equivalent for a space X.
i. X is a gb- $R_0$ space. ii. $x \in gbcl(\{y\})$ if and only if $y \in gbcl(\{x\})$ for any two points x and y in X.

**Proof.** (1) $\Rightarrow$ (2): Let $x \in gbcl(\{y\})$ and U be any gb- open set such that $y \in U$. Then by hypothesis, $x \in U$. Therefore, every gb- open set containing y contains x. Hence $y \in gbcl(\{x\})$.
(2) $\Rightarrow$ (1): Let V be a gb- open set such that $x \in V$. If $y \notin V$, then $x \notin gbcl(\{y\})$ and hence $y \notin gbcl(\{x\})$. Thus $gbcl(\{x\}) \subseteq V$. Therefore X is gb- $R_0$ space.
Next, we introduce the concept of gb- kernel of a set and utilizing it to characterize the notions of gb- $R_0$ and gb- $R_1$.

**Definition 3.5.** If X is a topological space and $A \subset X$. Then the gb- kernel of A (simply, gbKer(A)) is defined to be the set *gbKer*(*A*)= $\cap \{U \in GBO(X) : A \subseteq U\}$.

**Proposition 3.6.** If X is a topological space and x is any point in X. Then $y \in gbKer(\{x\})$ if and only if $x \in gbcl(\{y\})$

**Proof.** Suppose $y \notin gbKer(\{x\})$. Then there is a a gb- open set such that x belongs to V and $y \notin V$. Thus, we have $x \notin gbcl(\{y\})$. Similarly, we can prove the converse.
A subset $N_x$ of a topological space X is said to be a *gb- neighborhood* of a point x$\in$ X if there exists a gb- open set U such that $x \in U \subseteq N_x$.
It is easy to prove that,

**Proposition 3.7.** Let U be a gb- open subset of X, then U is a gb- neighborhood of each of its points.

**Definition 3.8.** We say That the family GBO(X) has property ( $\vartheta$ ) if the union of any collection of subsets belong to GBO(X) is in GBO(X).

**Theorem 3.9.** Let X be a space and A a subset of X and GBO(X) has property ( $\vartheta$ ). Then $gbKer(A) = \{x \in X : gbcl(\{x\} \cap A \neq \phi\}$.

**Proof.** Let $x \in gbKer(A)$ and $gbcl(\{x\}) \cap A = \phi$. Hence $x \notin (gbcl(\{x\}))^c = V$. So, by assumption, V is a gb- open set such that $A \subseteq V$. This is impossible, since $x \in gbKer(A)$.

Conversely, let $x \in X$ such that $gbcl(\{x\} \cap A \neq \phi$. Suppose that $x \notin gbKer(A)$. Then, there is a gb- open set U such that $A \subseteq U$ and $x \notin U$. Let $y \in gbcl(\{x\}) \cap A$, Thus U is a gb- neighborhood of y such that $x \notin U$, which is a contradiction. Hence $x \in gbKer(A)$.

**Theorem 3.10.** Let x, y be any two points X, if $gbcl(\{x\}) \neq gbcl(\{y\})$, then $gbKer(\{x\}) \neq gbKer(\{y\})$. If GBO(X) has property ( $\vartheta$ ), then the converse is true.

**Proof.** Suppose that $gbcl(\{x\}) \neq gbcl(\{y\})$. Then there is a point z in X such that $z \in gbcl(\{x\})$ and $z \notin gbcl(\{y\})$. So there is a gb- open set U containing z and hence containing x but not y, by Proposition 2.7. Thus $y \notin gbKer(\{x\})$. Therefore $gbKer(\{x\}) \neq gbKer(\{y\})$.

Conversely, Assume that $gbKer(\{x\}) \neq gbKer(\{y\})$. Then there exists a point z in X such that $z \in gbKer(\{x\})$ but $z \notin gbKer(\{y\})$. Since $z \in gbKer(\{x\})$, then, by Theorem 3.9, $\{x\} \cap gbcl(\{z\}) \neq \phi$ and hence $x \in gbcl(\{z\})$. Since $z \notin gbKer(\{y\})$, then $\{y\} \cap gbcl(\{z\}) = \phi$. And since $x \in gbcl(\{z\})$ and by assumption gbcl({z}) is gb- closed set, so $gbcl(\{x\}) \subset gbcl(\{z\})$ and $\{y\} \cap gbcl(\{z\}) = \phi$. Hence $gbcl(\{x\}) \neq gbcl(\{y\})$.

**Theorem 3.11.** For a topological space X. If GBO(X) has property ($\vartheta$). Then the following statements are equivalent.
i. X is a gb- $R_0$ space.
ii. For any $x \in X$, $gbcl(\{x\}) \subset gbKer(\{x\})$.
iii. For any gb- closed set F and a point $x \notin F$, there exists a gb- open set U such that $x \notin U$ and $F \subset U$.
iv. If F is a gb- closed set, then $F = \cap\{U \in GBO(X) : F \subseteq U\}$.
v. If F is a gb- closed set and $x \notin F$, then $gbcl(\{x\}) \cap F = \phi$.

**Proof.** i. $\Rightarrow$ (2): For $x \in X$, $gbKer(\{x\}) = \cap\{U \in GBO(X) : x \in U\}$. Since X is gb- $R_0$, so $gbcl(\{x\}) \subset U$ for any gb-open set U containing x. Therefore $gbcl(\{x\}) \subset gbKer(\{x\})$.
ii. $\Rightarrow$ (3): Assume that F is a gb- closed set and $x \in X$ such that $x \notin F$. Then for $y \in F$ we have $gbcl(\{y\}) \subset F$ and hence $x \notin gbcl(\{y\})$. So $y \notin gbcl(\{x\})$. Then there is a gb- open set $\upsilon$ containing y but not x for every $y \in F$. Let $U = \cup\{\upsilon \in GBO(X) : y \in \upsilon, x \notin \upsilon\}$, Then by assumption U is gb- open such that $x \notin U$ and $F \subset U$.
iii. $\Rightarrow$(4): Let F be any gb- closed set and $\omega = \cap\{U \in GBO(X) : F \subseteq U\}$. Then $F \subset \omega$ …..(*). Let $x \notin F$, then by (3) there is $U \in GBO(X)$ such that $x \notin U$ and $F \subset U$. Hence $x \notin \omega$. Hence $\omega \subset F$ …..(**). From (*) and (**) we have (4).
iv. $\Rightarrow$(5): If F is a gb- closed set where $x \notin F$. Then by (4). $x \notin \cap\{U \in GBO(X) : F \subseteq U\}$. So there is a gb- open set $H$ such that $x \notin H$ and $F \subset H$. Then $x \in H^c = M \subset F^c$, hence $gbcl(\{x\}) \subset M \subset F^c$. Therefore $gbcl(\{x\}) \cap F = \phi$.
V. $\Rightarrow$(1): If U is a gb- open set and x belongs to U. Then $U^c$ is gb- closed and $x \notin U^c$. Hence by (5), $gbcl(\{x\}) \cap U^c = \phi$. Thus $gbcl(\{x\}) \subset U$. Therefore X is gb- $R_0$ space.

## Generalized b- $T_0$ Spaces and Generalized b- $T_1$ Spaces

**Definition 4.1**. A space X is (1) generalized b- $T_0$ (briefly gb- $T_0$) if to each two distinct points x, y of X, there is a gb- open set containing one point but not the other.
(2) generalized b -$T_1$ (briefly gb- $T_1$) if to each two points x, y of X, there are a pair of gb- open sets, one containing x but not y and the other containing y but not x.

**Remark 4.2.** Every b- $T_1$ space is a gb- $T_1$ since every b-open set is gb- open. But not conversely as y the following example shows.

**Example 4.3.** Let X= {a, b, c} and $\tau$ = {X, $\phi$, {a}}. Then BO(X) = {X, $\phi$, {a},{a, c}, {a, b}} and GBO(X) = {X, $\phi$, {a}, {b}, {c}, {a, b}, {a, c}}, Then X is gb- $T_1$ but not b- $T_1$.

**Theorem 4.4.** Every topological space X is gb- $T_0$.

**Proof.** Let x and y be any two points in X.such that x≠ y. If $\operatorname{int}\{x\} \neq \phi$, then {x} is open, hence {x} is gb-open. Thus X is gb- $T_0$. Now, if $\operatorname{int}\{x\} = \phi$, then {x} is preclosed, thus X−{x} is a preopen. But every preopen set is b- open, so X−{x} is gb- open. Therefore X is gb- $T_0$.

**Theorem 4.5.** In any topological space *X*, gb- closures of distinct points are distinct.

**Proof.** Let x, y be any two points in X. such that x≠ y. Since every space is gb- $T_0$, by Theorem 4.4. There is a gb- open set U containing x or y, say x but not y. Then *X- U* is a gb- closed contains y but does not contain x. Since gbcl({*y*}) ⊆ *X- U*, so $x \notin gbcl(\{y\})$ . Therefore $gbcl(\{x\}) \neq bgcl(\{y\})$ .
Combining Theorem 3.10 and Theorem 4.5, we have the following result.

**Corollary 4.6.** For any two distinct points x, y in a topological space X, $gbKer(\{x\}) \neq gbKer(\{y\})$ .

**Definition 4.7.** A point x is called a gb-limit point of A if for each gb-open set *U* containing x, we have $(U - \{x\}) \bigcap A \neq \phi$ . The set of all *g*b-limit points of *A* is called to be *gb*-derived set of *A* (simply *gbd*(*A*)).

**Proposition 4.8.** Let X be a topological space and A be a subset of X, then gbcl(A)=A∪ gbd(A).

**Proof.** Obvious.

**Definition 4.9.** A topological space X is gb- symmetric if for any two points x and y in X, $x \in gbcl(\{y\})$ implies $y \in gbcl(\{x\})$ .

**Theorem 4.10.** If X is a gb- symmetric space. Then X is gb- $T_1$.

**Proof.** Let x, y be such that x≠ y . Since X is gb- $T_0$, there is a gb- open set U Such that $x \in U \subset X - \{y\}$ . Hence $x \notin gbcl(\{y\})$ . Since X is gb- symmetric, so $y \notin gbcl(\{x\})$ . Therefore there is a gb- open set V such that $y \in V \subseteq X - \{x\}$ . Hence X is gb- $T_1$.

**Definition 4.11.** A topological space X is called b- symmetric[13] if for any two points x and y of X, $x \in bcl(\{y\})$ implies $y \in bcl(\{x\})$ .

**Theorem 4.12.** Let X be a b- symmetric topological space then every singleton subset {x} of X is gb- closed.

**Proof.** Assume that $\{x\} \subseteq U \in \tau$ and $bcl(\{x\}) \not\subset U$ . Then $bcl(\{x\}) \cap U^c \neq \phi$. Let $y \in bcl(\{x\}) \cap U^c$ . Then $x \in bcl(\{y\}) \subseteq U^c$ , so x∉ U, this is a contradiction. Hence {x} is gb- closed.

**Theorem 4.13.** Let X be a topological space such that every singleton of X is gb- closed. Then X is gb-$T_1$.

**Proof.** Suppose that {a} is gb- closed for every a∈ X. Let x, y∈ X be distinct. Hence $\{x\}^c$ is a gb- open containing y but not x. Similarly $\{y\}^c$ is a gb- open set containing x but not y. Therefore X is a gb- $T_1$ Space.

The converse of the above theorem is not true in general as shown by the following example.

**Example 4.14.** In Example 4.3, X is gb- $T_1$ but {a} is not gb- closed.
In the next result, we provide a condition under which the converse of Theorem 4.13 is true.

**Theorem 4.15.** Let X be a topological space such that GBO(X) has property $(\vartheta)$ . If X is gb-$T_1$, then every singleton subset of X is gb- closed

**Proof.** Suppose that X is gb-$T_1$ and x is any point in X. Let y∈ $\{x\}^c$. Then $x \neq y$ . So there exists a gb- open set $U_y$ such that $y \in U_y$ but $x \notin U_y$ . Thus for each y∈ $\{x\}^c$, there exists a gb- open set $U_y$ such that $y \in U_y \subseteq \{x\}^c$ . Therefore $\cup\{y : y \neq x\} \subseteq \cup\{U_y : y \neq x\} \subseteq \{x\}^c$ which implies that $\{x\}^c \subseteq \cup\{U_y : y \neq x\} \subseteq \{x\}^c$ . Therefore $\{x\}^c = \cup\{U_y : y \neq x\}$ . Since $U_y$ is gb- open in X, by assumption. Hence $\{x\}^c$ is gb- open and so {x} is gb- closed.

Combining Theorems 4.13 and 4.15, we have the following result.

**Corollary 4.16.** If GBO(X) has property ( $\vartheta$ ), then X is gb-$T_1$ if and only if X is b- symmetric.

**Theorem 4.17.** The following statements are equivalent for any space X.
i. X is gb- $R_0$, ii. X is gb- $T_1$.

**Proof.** (2) $\Rightarrow$ (1): Obviuos.
(1) $\Rightarrow$ (2): This is hold since gb- $T_1$ is equivalent to gb- $R_0$ and gb- $T_0$.
Now, we provide conditions under which the property of being a bg- $T_1$ is invariant.

**Theorem 4. 18.** Let *f*: $(X, \tau) \longrightarrow (Y, \sigma)$ be an injective, gb- irresolute mapping, then X is gb- $T_1$ if Y is gb- $T_1$.

**Proof.** Obvious.

**Theorem 4.19.** The property of being a gb- $T_1$ space is preserved by pre- gb- open, bijective mapping and hence it is gbc- topological property.

**Proof.** Let X be a gb- $T_1$ space and Y be any space. Let f be a one- one pre- gb- open mapping of X onto Y. Let $y_1$ and $y_2$ be any two distinct point of Y. Since f is bijective, there exist distinct points $x_1$ and $x_2$ of X such that $y_1$= f($x_1$) and $y_2$= f($x_2$). But X is gb- $T_1$ space, so there exist gb- open sets U and V such that $x_1 \in U$ but $x_2 \notin U$ and $x_2 \in V$ but $x_1 \notin V$. Hence $y_1 \in f(x_1) \in f(U)$ but $y_2 \in f(x_2) \notin f(U)$ and $y_2 \in f(x_2) \in f(V)$ but $y_1 \in f(x_1) \notin f(V)$. Since f is pre- gb- open , it follows that *f(U)* and *f(V)* are gb- open subsets of Y such that $y_1 \in f(U)$ but $y_2 \notin f(U)$ and $y_2 \in f(V)$ but $y_1 \notin f(V)$. Hence Y is a gb- $T_1$.

## gb- $T_2$ Topological Spaces.

**Definition 5.1.** A space X is said to be gb- $T_2$, if for each pair of distinct points x, y of X, there exist disjoint gb- open sets U, V such that x$\in$ U and y$\in$ V.
It is easy to see that,

**Remark 5.2.** (1) b- $T_2$ $\Rightarrow$ gb- $T_2$, (2) gb-$T_2$ $\Rightarrow$ gb- $T_1$.
The converse of each part is not true in general.

**Example 5.3.** In Example 4.3, X is gb- $T_2$ but not b- $T_2$.

**Problem 5.4.** Give an example to show that the converse of (2) is not true in general.

**Theorem 5.5.** For a space X, the following statement are equivalent.
i. X is gb- $T_2$,, ii. Let $x_0$ be a point in X, then for any $x \in X$ , $x \neq x_0$ , there is a gb- open set U in X containing $x_0$ such that $x \notin gbcl(U)$ , iii. For each $x \in X$ , $\cap${gbcl(U): U is gb- open in X containing x }={x}.

**Proof.** (1)$\Rightarrow$(2): Let $x_0 \in X$ be g iven and consider $x \neq x_0$ . Since X is a gb- $T_2$ space, there exist disjoint gb- open sets U and V containing $x_0$ and x respectively. Then $V^c$ is gb- closed, gbcl(U)$\subseteq V^c$. Thus $x \in V^c$ , a contradiction. Hence $x \notin gbcl(U)$ .
(2) $\Rightarrow$ (3) For each $x \neq y$ , there exist a gb- open set U such that x$\in$ U and $y \notin gbcl(U)$ , So $\cap${gbcl(U): U is gb- open in X and x$\in$ U}= {x}.
(3)$\Rightarrow$(1): Let $x \neq y$ , then $y \notin \cap\{gbcl(U) : U \in GBO(X), x \in U\} = \{x\}$ . Hence there is a gb- open set $V_y$ containing y such that $V_y \cap U = \phi$ . Therefore X is gb- $T_2$.
We introduce the following characterization of gb- $T_2$.

**Theorem 5.6.** Let X be a space and GBO(X) has property ( $\vartheta$ ). Then X is gb- $T_2$ if and only if X is gb- $R_1$.

**Proof.** *Necessity***.** If $x, y \in X$ such that $x \neq y$, then by Theorem 4.5, $gbcl(\{x\}) \neq gbcl(\{y\})$. Since X is gb- $T_2$, there exist disjoint gb- open sets U, V such that $x \in U$ and $y \in V$. Since X is gb- $T_2$, so by assumption and Theorem 4.15, every singleton is gb- closed. Hence $gbcl(\{x\}) = \{x\} \subseteq U$ and $gbcl(\{y\}) = \{y\} \subseteq V$. Thus X is gb- $R_1$.

*Sufficiency*. Let $x, y \in X$ such that $x \neq y$. Then, by Theorem 4.5, $gbcl(\{x\}) \neq gbcl(\{y\})$. By assumption, there are disjoint gb- open sets U and V such that $x \in gbcl(\{x\}) \subseteq U$ and $y \in gbcl(\{y\}) \subseteq V$. So X is gb- $T_2$.
The following result is a characterization of gb- $R_1$ spaces.

**Theorem 5.7.** Let X be gb- $R_1$ topological space, then for any $x, y \in X$ such that $gbcl(\{x\}) \neq gbcl(\{y\})$, there exist gb- closed sets U and V such that $x \in U$, $y \notin U$, $y \in V$, $x \notin V$ and $X = U \cup V$. And if GBO(X) has property ($\vartheta$), then the converse holds.

**Proof.** Let $x, y \in X$ such that $gbcl(\{x\}) \neq gbcl(\{y\})$, hence $x \neq y$. Then by hypothesis, there exist disjoint gb- open sets U and V such that $x \in gbcl(\{x\}) \subseteq U$ and $y \in gbcl(\{y\}) \subseteq V$. Let F= $U^c$ and H= $V^c$, then F, H are gb- closed sets such that $x \in H$, $y \notin H$ and $y \in F$, $x \notin F$ and $X = F \cup H$.

Conversely, Suppose that x and y are distinct points of X, such that $gbcl(\{x\}) \neq gbcl(\{y\})$. By hypothesis, there are gb- closed sets F and H such that $x \in F$, $y \notin F$, $y \in H$, $x \notin H$ and $X = F \cup H$. Set U= $H^c$ and V=$F^c$, then U, V are gb- open and $x \in U$, $y \in V$ and $U \cap V = \phi$. Therefore X is gb- $R_1$, by Theorem 5.6.
The property of being gb- $T_2$ is invariant under gb- irresoluteness, injective mappings.

**Theorem 5.8.** Let *f*: $(X, \tau) \longrightarrow (Y, \sigma)$ be gb- irresolute and injective. If Y is gb- $T_2$, then X is gb- $T_2$.

**Proof.** Obvious.
Similar to Theorem 4.19 we obtain,

**Theorem 5.9.** The property of being a gb- $T_2$ space is preserved by pre- gb- open, bijective mapping and hence gbc- topological property.

## Applications

The digital line[14, 15] or the so called Khalimsky line $(Z, \kappa)$ is the set Z of integers with the topology $\kappa$ having $\{\{2n-1, 2n, 2n+1\} : n \in Z\}$ as a subbase. It is proved by Dontchev and Ganster[4] that the digital line $(Z, \kappa)$ is a $T_{3/4}$ space, which fails to be $T_1$.

In this section we give a new characterization of $T_{3/4}$ by utilizing b-$T_1$ spaces. Also we investigate some separation axioms for the digital line.

**Theorem 6.1.** Every $T_{3/4}$ space is b-$T_1$.

**Proof.** Since X is $T_{3/4}$, so by Proposition 2.16, every singleton is closed or regular open. Since every regular open is semi- closed. Hence every singleton is closed or semi- closed. But every closed and every semi- closed is b- closed. Hence every singleton is b- closed. Therefore X is b- $T_1$, by Proposition 2.15.

Dontchev and Ganster[4] proved that every $T_{3/4}$ is $T_{1/2}$. We provide a new characterization for $T_{3/4}$ space.

**Theorem 6.2.** The following statements are equivalent for any space X.
i. X is $T_{3/4}$, ii. X is $T_{1/2}$ and b-$T_1$.

**Proof.** (1) $\Rightarrow$ (2): By Theorem 6.1, and the fact that every $T_{3/4}$ is $T_{1/2}$, we have (2).

(2) $\Rightarrow$(1): Since X is $T_{1/2}$, so every singleton {x} is either open or closed. And since X is b-$T_1$, so every singleton is b- closed and so $\text{int}(cl(\{x\})) \cap cl(\text{int}(\{x\})) \subseteq \{x\}$. Then, if {x} is open, we have $\text{int}(cl(\{x\})) = \{x\}$. Thus {x} is regular open for any x$\in$ X. Therefore by Proposition 2.16, X is $T_{3/4}$.
Combining Remarrk 2.14 (3) and Theorem 6.2, we get the next result which is (Theorem 4.14) of Dontchev and Ganster[4].

**Corollary 6.3.** If X is both $T_{1/2}$ and s-$T_1$, then X is $T_{3/4}$.

**Corollary 6.4.** The digital line (Z, $\kappa$) is b- $T_1$ and hence gb- $T_1$.
Caldas and Jafari[13] have introduced the following definition and Lemma.

**Definition 6.5.** A space X is said to be a
i. b- $R_0$ space if every b- open set contains the b- closure of each of its singletons. ii. b- $R_1$ space if for x, y in X with $bcl(\{x\}) \neq bcl(\{y\})$, there exist disjoint b- open sets U and V such that such that $bcl(\{x\})$ is a subset of U and $bcl(\{y\})$ is a subset of V.

**Lemma 6.6.** For any space X, b- $R_1$ and b- $T_2$ are equivalent.

**Theorem 6.7.** The digital line is b- $T_2$, b- $R_1$ and b- $R_0$.

**Proof.** It is proved by Fujimoto and Takigawa[16] that (Z, $\kappa$) is s- $T_2$ and hence b- $T_2$, by Remark 2.14 (2). Hence by Lemma 6.6, it is b- $R_1$ and hence b- $R_0$.

**Corollary 6. 8.** The digital line is gb- $T_2$ and hence gb- $T_1$.

**Proof.** Follows from Theorem 6.7 and Remark 5.2.
Next, we give another proof of the fact that digital line is b- $R_1$.

**Theorem 6.9.** For any point x of (Z, $\kappa$), bcl{x}= {x}.

**Proof.** For any point x of (Z, $\kappa$), it known[17] that:
i. If x= 2t, $t \in Z$, then cl({x})= {x} and int({x})= $\phi$,
ii. If x= 2t+1, $t \in Z$, then cl({2t+1})={2t, 2t+1, 2t+2} and int({2t+1})= {2t+1}.
Now, if x= 2t, then bcl{x}= bcl{2t}= $\{2t\} \cup [cl(\text{int}(\{2t\})) \cap \text{int}(cl(\{2t\}))] = \{2t\}$.
And if x= 2t+ 1, then bcl{x}= bcl{2t+1}= $\{2t+1\} \cup [cl(\text{int}(\{2t+1\})) \cap \text{int}(cl(\{2t+1\}))] = \{2t+1\}$.

**Theorem 6.10**. The digital line is b- $R_1$.

**Proof.** Let p and q be two points of (Z, $\kappa$) such that bcl{p}$\neq$bcl{q}. Hence $p \neq q$. We have the following cases:
i. If p= 2t and q= 2s where $t \neq s$ and $t < s$. Let U = {2t- 1, p} and V= {q, 2s+1}.
Then U and V are disjoint b- open sets containing p= bcl({p}) and q= bcl({q}) respectively.
ii. If p= 2t and q= 2s+1, where $t < s$. Let U= {2t- 1, p} and V= {q, 2s+2}. Then
U and V are disjoint b- open sets containing p= bcl({p}) and q= bcl({q}) respectively.
iii. If p= 2t+ 1 and q= 2s+ 1 where $t < s$. Let U= {2t, p} and V= {q, 2s+2}. Then U and V are disjoint b- open sets containing p= bcl({p}) and q= bcl({q}) respectively.
Hence (Z, $\kappa$) is b- $R_1$.

## Conclusion

The class of generalized closed sets has an important role in general topology, especially its suggestion of new separation axioms which are useful in digital topology. In this work we introduced and study new types of separation axioms namely, gb- $R_i$, i= 1, i= 0, 1, and gb- $T_i$, i= 1,2. Several characterizations and properties of these concepts are provided. A new characterization of $T_{3/4}$ space is investigated. We proved that Khalimsky line, or digital line (Z, $\kappa$) is b- $T_2$, b- $R_1$, b- $R_0$, gb- $T_1$ and gb- $T_2$.

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