Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. **1(8),** 8-15, September **(2013)** *Res. J. Mathematical & Statistical Sci.*

A Mathematical Study of two phases Pulmonary Blood Flow in Lungs with special References to Lungs Infection Copd

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Available online at: www.isca.in
Received 23rd May 2013, revised 22nd July 2013, accepted 3rd August 2013

Abstract

In the present paper, we formulate the pulmonary blood flow in Lung. Keeping in view the nature of pulmonary circulatory system in human body, the viscosity increases in the arterioles due to formation of Roulex along axis by red blood cells, P.N. Pandey and V. Upadhyay have considered the blood flow of two phase, one of which is that of red blood cells and other is plasma. They have also applied the Herschel-Bulkley non- Newtonian model in bio fluid mechanical set-up .We have collected clinical data in case of COPD for Hematocrit v/s Blood pressure . The graphical presentation for particular parametric value in much closed to the clinical observation. The overall presentation is in tensorial form and solution technique adapted is analytical as well as numerical. The role of Hematocrit is explicit in the determination of blood pressure in case of pulmonary disease-COPD.

Keywords: Structure of the lungs, circulatory system, COPD, Herschel-Bulkley model etc.

Introduction

Structure and function of the lungs: The lungs are paired organs in the chest. Each human has two lungs. The right lung has three lobes, but the left lung has two lobes. It is smaller to accommodate the heart, which is positioned below the left lung and takes up the space of the third lobe in the chest cavity. Air enters the lung via the bronchial tree, a series of smaller branches off of the windpipe (or trachea) off the windpipe, the left bronchus branches into the left lung, and the right bronchus branches into the right lung. These bronchi then branch into bronchioles, which terminate at the alveolar sacs. The alveolar sacs contain the small, thin- walled air pouches known as alveoli. The alveoli are the smallest units of the lung tissue. The lungs behave just like purification station for blood. Lungs bring oxygen to the blood stream or tiny sacs in the lungs. When oxygen enters the blood, hemoglobin picks it up and transports it throughout the body. The lungs also remove carbon dioxide from the blood stream.

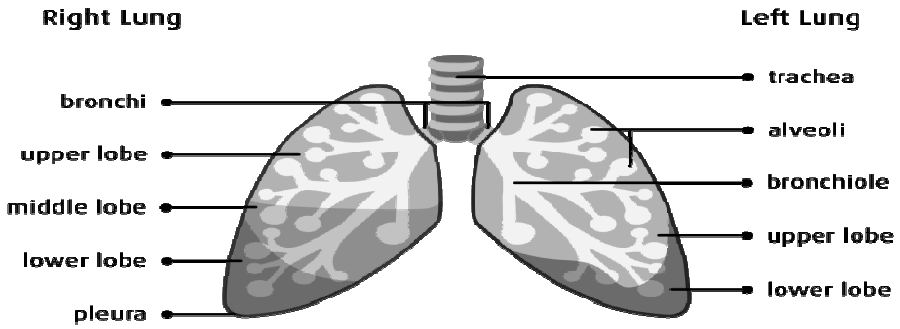

Figure-1
Schematic diagram of the Lungs

Structure and function of the circulatory system: The human circulatory system is really a two-part system, whose purpose is to bring oxygen-bearing blood to all the tissues of the body. The circulatory system is made up of the vessels and the muscles that help and control the flow of the blood around the body. This process is called circulation. The main parts of the system are the heart, arteries, capillaries, and veins. The circulatory system is a network that carries blood throughout the body. The circulatory system also helps regulate body temperature and carries substances that protect the body from diseases. In addition, the system transport chemical substances called hormones, which help regulate the activities of various part of the body.

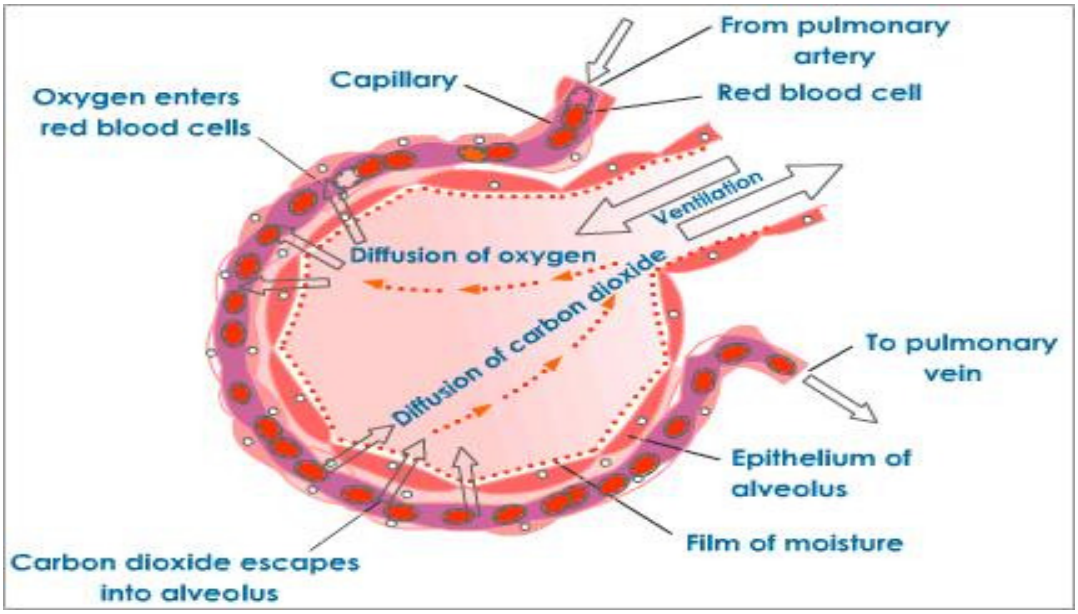

Figure-2
Flowchart of the circulatory system

Description of Chronic Obstructive Pulmonary Disease (COPD)

What is chronic obstructive pulmonary disease (COPD)? Chronic obstructive pulmonary disease is finally gaining the attention needed to begin to solve this common problem. It is defined as chronic airflow obstruction that is progressive and only partly reversible (Pauwels et al 2001; Global Initiative for Chronic Obstructive Lung Disease 2004). Today, COPD is the fourth most common cause of death in the USA, and is the only disease state that is rising in morbidity and mortality amongst the top five killers. Chronic obstructive pulmonary disease (COPD) is a lung disease that makes it hard to breathe. It is caused by damage to the lungs over many years, usually from smoking. COPD is often a mix of two diseases:

Chronic bronchitis: In chronic bronchitis, the airways that carry air to the lungs get inflamed and make a lot of mucus. This can narrow or block the airways, making it hard for you to breathe.

Emphysema: In a healthy person, the tiny air sacs in the lungs are like balloons. As you breathe in and out, they get bigger and smaller to move air through your lungs. But with emphysema, these air sacs are damaged and lose the stretch. Less air gets in and out of the lungs, which makes you feel short of breath.

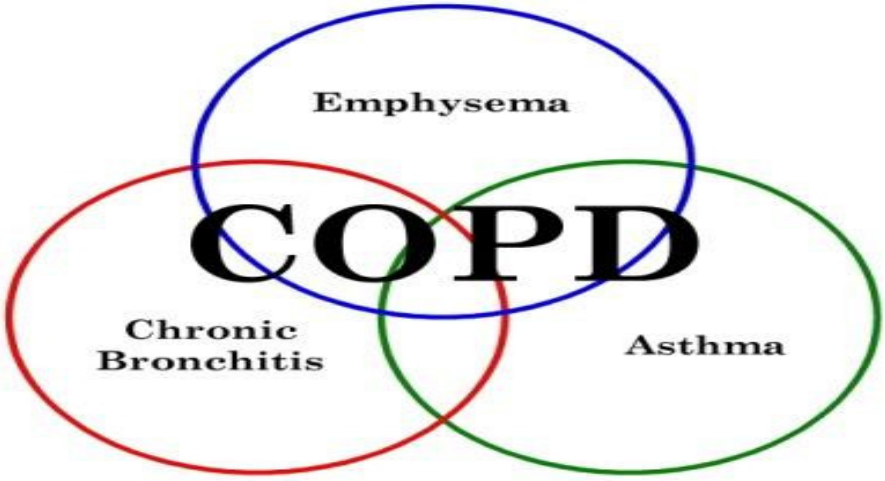

Figure-3

Causes: Smoking is the primary cause of COPD. It can also be caused by exposure to pollutants or toxic chemical. One rare form of COPD is inherited.

See risk factors: i. Smoking the longer you smoke and the more packs of cigarettes you smoke, the higher your risk. People who smoke pipes and cigars, and those who are exposed to large amounts of secondhand smoke, also have greater risk. ii. Genetics people with a rare hereditary disorder called alpha – 1 anti – trypsin deficiency lack an enzy that helps protect the lungs from damage. iii. Being over age 50. iv. Working around industrial smoke, excessive dust, or other air pollutants (for example, miners, furna workers, and grain farmers).

Prevent Care: i. If you smoke, quit, ii. If you have COPD, avoiding respiratory infections is very important. iii. Eating foods rich in antioxidants, magnesium and other minerls, and omega – 3 fatty acids (including fruits, vegetables, and fish) may help lower your risk for COPD.

Real Model (assumption of approaching reality): Kumar VH, Ryan RM said that role of these growth factor and their cellular interactions in Broncho pulmonary Dysplasia and in tissue repair following lung injury may lead to development of better therapeutic modalities in treating these disorders [1]. Puja K. Mehta and Kathy K. Griendling suggested focus on the structure and function of AT1 receptors and the major signaling mechanism by which angiotension influences cardiovascular physiology and pathology [2]. Debra S, Faffe, Walter A. Zin focus about this review that mechanical properties of lung tissue and how the stress-bearing elements of lung parenchyma our influence its behavior [3]. Dennis Wilken, Marcial V Garrido, Ulf Manuwald and Xaver Baur gave their views that a asbestos exposure is related to restrictive and obstructive lung function impairment [4]. J. N. Oko-ose, V. Iyawe, E. Egbagbe, M. bomoyi, Cannie C.W. Hsia and Merryn H. Tawhai discuss that lung function differs significant in subject with SCD (Sickle cell disease) compared with matched controls of a similar age and gender[5].

Constitution of blood: According to I.W. Sherman and V.G. Sherman [6], blood is the mixed fluid. Mainly there are two phases in blood. First plasma and other blood cells. The components of blood include red blood cells, white blood cells, platelets, and plasma.

The red blood cells, which are called erythrocytes have the important responsibility of carrying the oxygen throughout the body. Packed with hemoglobin (an iron-bearing protein). RBC is homogeneous distribution in plasma.

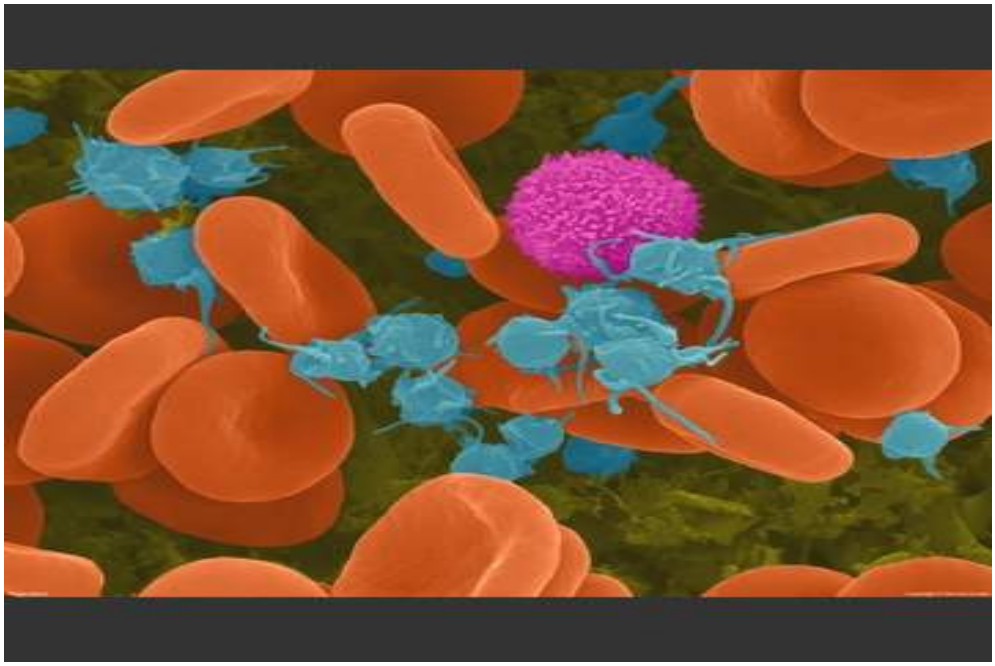

Figure-4

The white blood cells (purple), which are also called leukocytes, are involved in functions controlled by the immune system. The immune system is responsible for fighting infections. If a person has a low white blood cell count, it means that the immune system is not functioning properly. If a white blood cell count is too high, it indicates that the person has some type of infection.

Platelets (blue) are pieces of cells that work to form blood clots. They work to keep your body from losing too much blood when you sustain an injury and help in wound healing. Plasma is the "stream'' in blood stream. The plasma contains many important proteins, without which you would die.

Mathematical Modeling: J.C. Mishra and S.K. Pandey have suggested that blood flow in vessels is a peristaltic transport system because they thought blood is having two layers of fluid while in the peripheral reasons of vessels blood flow is a Newtonian phenomena. Blood is in the liquid form and it is non-newtonian. Though blood is not an ideal fluid, even to develop the equation of motion. we start with a model of ideal fluid. The second important principle of fluid dynamics is that of conservation of momentum. The equation of motion is based on this principle. According to this principle, the total momentum of any fluid system is conserved in absence of external force. The blood can be considered as a homogeneous mixtures of two phases. We derive the fundamental equation of continuity,which is a mathematical expression of principal of conservation of matter.

Equation of Continuity: The flow of blood is affected by the presence of blood cells. This effect is directly proportional to the volume occupied by blood cells. Let the volume portion covered by blood cells in unit volume be *X* ,this *X* is replaced by 1*/100*, where *H* is the Hematocrit the volume percentage of blood cells. Then the volume portion covered by the plasma will be (*1-X*). If mass ratio of cells to plasma is r then clearly: $r = \frac{X\rho_c}{(1-X)\rho_p}$ (1)

where ρ_c and ρ_p are densities of blood cells and plasma respectively. Usually this mass ratio is not a constant. Even then this may be supposed to constant in present context.

The both phase of blood, i.e. blood cells and plasma move with the common velocity. Campbell and Pitcher have presented a model for this situation. According to this model, we consider the two phase of blood separately. Hence according to the principle of conservation of mass in pulmonary circulatory system, equation of continuity for two phases are as follow.

$\frac{\partial(X\rho_c)}{\partial t} + (X\rho_c V^i)_{,i} = 0$ (2)

And $\frac{\partial(1-X)\rho_p}{\partial t} + ((1-X)\,\rho_c\, V^i)_{,j} = 0$ (3)

Where *V* is the common velocity of two phase blood cells and plasma. If we define the uniform density of blood ρ_m as follows:

$\frac{1+r}{\rho_m} = \frac{r}{\rho_c} + \frac{1}{\rho_p}$ (4)

Then equation (2) and (3) can be combined together as: $\frac{\partial \rho_m}{\partial t} + (\rho_m v^i)_{,j} = 0$ (5)

Equation of motion for blood-flow: The hydro dynamical pressure *p* between the two phases of blood can be supposed to be uniform because the both phases i.e. blood cells and plasma is always in equilibrium state in blood. Taking viscosity coefficient of blood cells to be η_c and applying the principle of conservation of momentum in pulmonary circulatory system, we get the equation of motion for the phase of blood cells as follows:

$\frac{\partial V^i}{\partial t} + (X\rho c\, Vi)\, V^i_{,j} = -X\, p_{,j} g^{ij} + X\eta_c\, (g^{ij} v^i_{,k})_{,j}$ (6)

Taking the viscosity coefficient of plasma to be η_p, the equation of motion for plasma will be as follows :

$(1\text{-}X)\,\rho_p \frac{\partial V^i}{\partial t} + \{(1\text{-}X)\,\rho_p\, v^i\}\, V^i_{,j} = -(1\text{-}X)\, p_{,j}\, g^{ij} + (1\text{-}X)\,\eta_c\, (g^{ij} V^i_{,k})_{,j}$ (7)

Now adding equation 6 and 7 and using relation 4, the equation of motion for blood flow with the both phases will be as follows:

$\rho_m \frac{\partial V^i}{\partial t} + (\rho_m v^i) V^i_{,j} = -p_{,j}\, g^{ij} + \eta_m (g^{ij}\, V^i_{,k})_{,j}$ (8)

Where $\eta_m = X\,\eta_c + (1\text{-}X)\eta_p$ is the viscosity coefficient of blood as a mixture of two phases. In this situation, the blood cells line up on the axis to build up rolex. Hence a yield stress is produced. Though this yield stress is very small, even then the viscosity of blood is increased nearly ten times.

That's why the Herschel-Bulkley Law holds good on the two phase blood flow through veins arterioles, venules and whose constitutive equation is as follows: $T' = \eta_m e^n + T_p$ $(T' \geq Tp)$ *and*

Where T is the yield stress.

When strain rate $e=0 (T' < T_P)$ a core region is formed which flows just like a Plug. Let the radius of the plug be r_p . The stress acting on the surface of the plug will be T. Equating the forces acting on the plug, we get figure-3

Whose generalized form will be as follows:

$T^{ij} = -pg^{ij} + T_e^{ij}$

Where $T_e^{ij} = \eta_m (e^{ij})^n$ *while* $e^{ij} = (g^{jk} v^i_{,k} + g^{ik} v^j_{,k})$

$P 2\pi r_p^2 = T_P \, 2\pi r_p$

$\Longrightarrow \; r_P = \frac{2T_P}{P}$ (9)

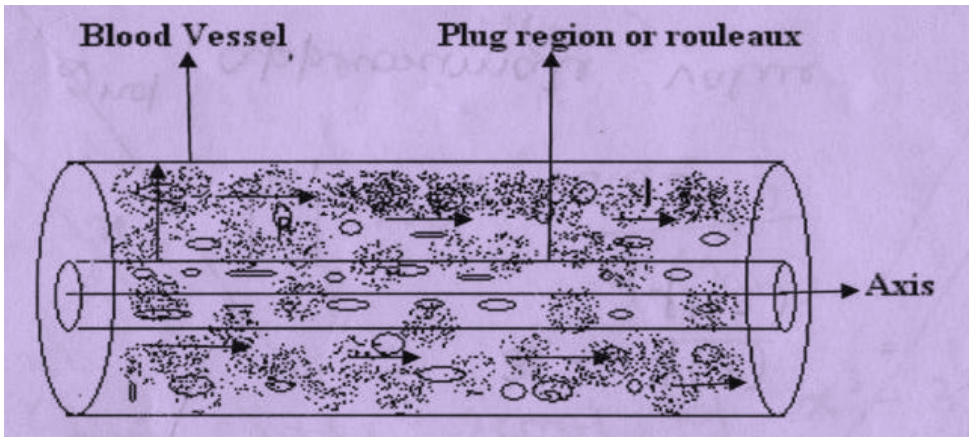

Figure-5
Herschel Bulkley blood flow

Where the symbols have their usual meanings.
Now we consider the basic equation for Herschel- Bulkley flow as follows:

Equation of continuity: $\frac{1}{\sqrt{g}} (\sqrt{g} v^i) = 0$ (10)

The equation of motion $\rho_m \frac{\partial v^i}{\partial t} + \rho_m \, v^j \, v^i_{,j} = -T^{ij}_{,j}$

Where all the symbols have their usual meanings.

Analysis

Since the blood vessels are cylindrical, the above governing equations are transformed into cylindrical form. As we know earlier $x^1 = r$, $x^2 = \theta$, $x^3 = z$

Matrix of metric tensor in cylindrical co-ordinates is as follows: $[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

While matrix of conjugate metric tensor is as follows: $[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Where as the Christoffel's symbols of 2nd kind as follows: $\left\{ \begin{matrix} 1 \\ 2 \end{matrix} \; \begin{matrix} \\ 2 \end{matrix} \right\} = -r \left\{ \begin{matrix} 2 \\ 2 \end{matrix} \; \begin{matrix} \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \; \begin{matrix} \\ 2 \end{matrix} \right\} = \frac{1}{r}$ remaining others are zero.

Relation between contra variant and physical components of velocity of blood flow will be as follows:
$\sqrt{g^{11}v^1} = v_r \Rightarrow v_r = v^1$, $\sqrt{g^{22}v^2} = v_\theta \Rightarrow v_\theta = rv^2$, $\sqrt{g^{33}v^3} = v_z \Rightarrow v_z = v^3$, Again the physical components of - $p_{,j}\, g^{ij}$ are - $\sqrt{g_{ij}}\, p_{,j} g^{ij}$

Now, equation 9 and 10 are transformed into cylindrical form so as to solve them solve as power law model to get
$\frac{dv}{dr} = \left(\frac{pr}{2\eta_m}\right)^{1/n}$

Replace r from r-r_p

$\frac{dv}{dr} = \left[\frac{p(r-r_p)}{2\eta_m}\right]^{\frac{1}{n}}$, $\frac{dv}{dr} = \left[\frac{\frac{1}{2}p(r-r_p)}{\eta_m}\right]^{\frac{1}{n}}$, $\frac{dv}{dr} = \left[\frac{\frac{1}{2}Pr-\frac{1}{2}Pr_P}{\eta_m}\right]^{\frac{1}{n}}$, $\frac{dv}{dr} = \left[\frac{\frac{pr}{2}-\frac{prp}{2}}{\eta_m}\right]^{\frac{1}{n}} \Rightarrow -\frac{dv}{dr} = -\left(\frac{p}{2\eta_m}\right)^{\frac{1}{n}} (r-r_p)^{\frac{1}{n}}$ (11)

Integrating above equation 11 under the no slip boundary condition v =0 at r =R, so as to get:

$V = \left(\frac{p}{2\eta_m}\right)^{\frac{1}{n}} \frac{n}{n+1}\left[(R-r_p)^{\frac{1}{n}+1} - (r-r_p)^{\frac{1}{n}+1}\right]$ (12)

This is the formula of the velocity of blood flow in arterioles, venules and veins. Putting r=r_p to get the velocity V_p of plug flow as follows:

$V_p = \frac{n}{n+1}\left(\frac{p}{2\eta_m}\right)^{\frac{1}{n}} (R-r_p)^{\frac{1}{n}+1}$ (13)

Where the value of r_p is taken from 7.

Results and Discussion (Biophysical Interpretation)

Observation: Hametocrit vs blood pressure from an authorized. G.S.V.M.medical college Kanpur by Dr. Avdhesh Kumar

Patient name: Mrs. Baby (age- 50 years old)

Diagnosis: COPD (Chronis obstructive pulmonary disease)

Date	HB(Hemoglobin)	B.P.(Blood Pressure)	3× hemoglobin =Hematocrit
04-o1-2013	15.2	130/80	3× 15.2 = 45.6
19-01-2013	14.6	110/80	3 × 14.6 = 43.8
29-01-2013	14.1	106/80	3 × 14.1 = 42.3
04-02-2013	13.6	120/90	3 × 13.6 = 40.8
07-02-2013	13.6	132/90	3 × 13.6 = 4o.8

The flow flux phased blood flow in arterioles, venules and veins is
$Q = \int_0^{r_p} 2\pi r v_p dr + \int_{r_p}^{R} 2\pi r v dr$, $= \int_0^{r_p} 2\pi r \frac{n}{n+1} (p/2\eta_m)^{1/n} (R-r_p)^{1/n+1} dr + \int_0^{r_p} 2\pi r \frac{n}{n+1} (p/2\eta_m)^{1/n}$, $[(R-r_p)^{1/n+1} - (r-r_p)^{1/n+1}]\, dr$
Using 12 and 14

$= \frac{2\pi n}{n+1} (P/2\eta_m)^{1/n} (R-r_p)^{1/n+1} \left[\frac{r^2}{2}\right]_0^{r_p} + \frac{2\pi n}{n+1}\left(\frac{p}{2\eta_m}\right)^{\frac{1}{n}} \left[\frac{r^2}{2}(R-r_p)^{\frac{1}{n}} - \frac{r(r-r_p)}{\frac{1}{n}+2} + \frac{(r-r_p)^{\frac{1}{n}+3}}{(\frac{1}{n}+2)(\frac{1}{n}+3)}\right]_{r_p}^{R}$

$= \frac{2\pi n}{(n+1)} (P/2\eta_m)^{1/n} r_P^2 (R-r_P)^{\frac{1}{n}+1} + R^2 (R-r_P)^{\frac{1}{n}+1} - \frac{2R(R-r_P)^{\frac{1}{n}+2}}{\frac{1}{n}+2} + \frac{2(R-r_p)^{\frac{1}{n}+3}}{(\frac{1}{n}+2)(\frac{1}{n}+3)} - r_P^2 (R-r_P)^{\frac{1}{n}+1}$

$= \frac{\pi n}{(n+1)} (P/2\eta_m)^{1/n} R^{\frac{1}{n}+3}\left[\frac{r_p^2}{R^2}(1-\frac{r_p}{R})^{\frac{1}{n}+1} + (1+\frac{r_P}{R})(1-\frac{r_P}{R})\frac{2(1-\frac{r_p}{R})^{\frac{1}{n}+2}}{(\frac{1}{n}+2)} + \frac{2(1-\frac{r_P}{R})^{\frac{1}{n}+3}}{(\frac{1}{n}+2)(\frac{1}{n}+3)}\right]$

P = Pressure gradient, v = Viscosity of mixture (Blood), n = Parameter

Now we have, Q **= 425** *ml/min,* R = 1, r_p= 1/3, η_m = 0.027 Pascal second, η_P = 0.0013 and H = 45.6 P = 130

We also know that, $\eta_m = \eta_c X + \eta_P (1- X)$,

Where $X = \frac{H}{100} = \frac{43.5}{100} = 0.435$

$\Rightarrow$ $0.027 = \eta_c(0.435) + (0.0013)(1-0.435)$
$\Rightarrow$ $\eta_c = 0.0453$

Again, $\eta_m = (0.0453)\frac{H}{100} + (0.0013)\left(1 - \frac{H}{100}\right)$, $\eta_m = (0.0453)(0.435) + (0.0013)(0.565)$ $\eta_m = 0.00177$

Solving by numerical method, we get n= 0.9825, P= ((0.000453) H+ 0.0039) $\left[1827.23 \times \frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}\right]^n$
Putting the value of n , we get
P= ((0.000453) H+ 0.0039) $\left[1827.23 \times \frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}\right]^{0.9825}$
P= ((0.000453)*H*+ 0.0039) (5637.18) (0.2868) at n = 0.9825

at H = 45.6 P = 39.780 ≈ 40
at H = 43.8 P = 38.38o ≈ 38
at H= 42.3 P = 37.2837 ≈ 37
at H= 40.8 P = 36.1865 ≈ 36
at H=40.8 P = 36.1865 ≈ 36

Hematocrit	40.8	40.8	42.3	43.8	45.6
Blood pressure	36.1865≈36	36.1865≈36	37.2837≈37	38.380≈38	39.780≈40

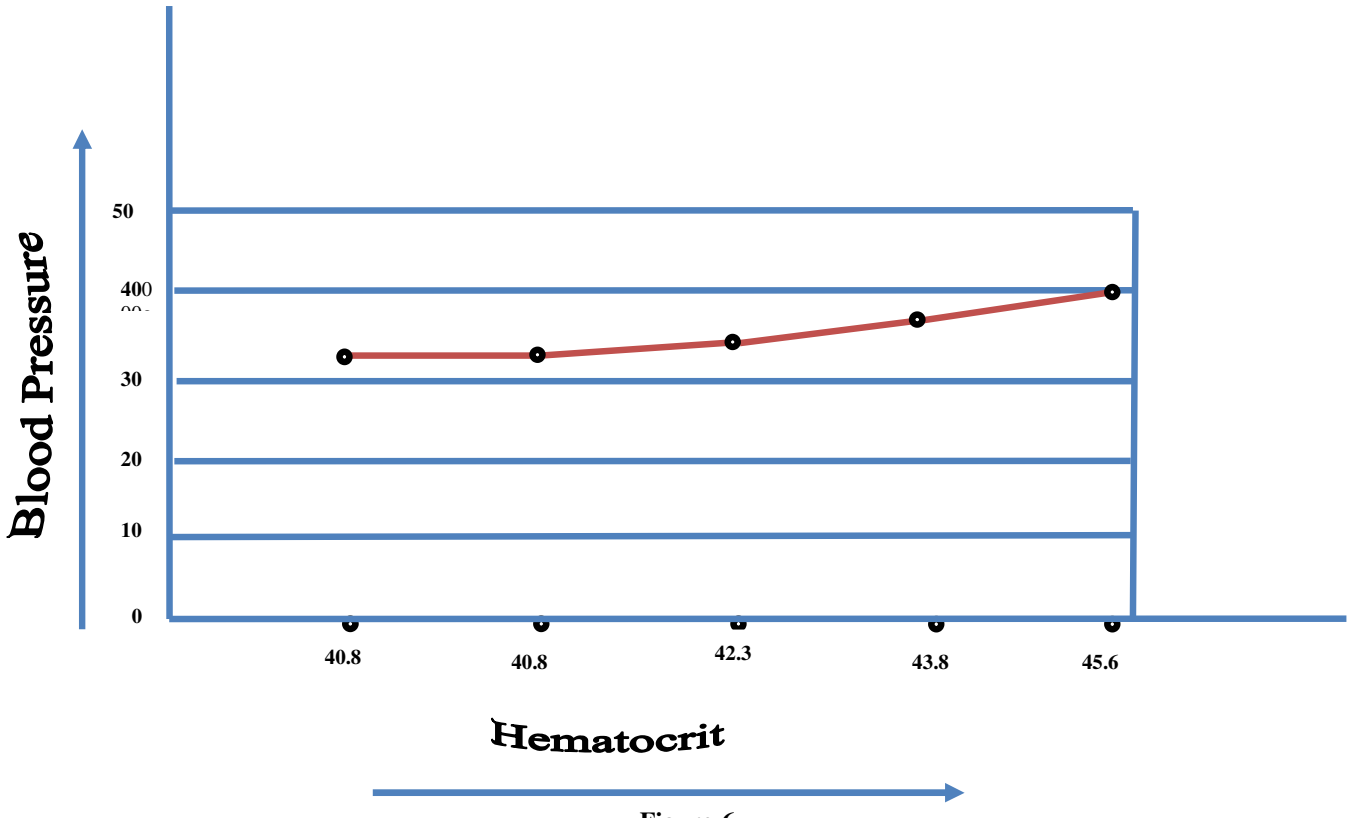

Figure-6

Conclusion

A simple survey of the graph between blood pressure and hematocrit in COPD (chronic obstructive pulmonary disease) patient shows that when Hematocrit is increased then blood pressure also increased. Hence Hematocrit is directly proportional to blood pressure. i.e. Hematocrit ∝ Blood pressure

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