Research Journal of Mathematical and Statistical Sciences ______________________________ISSN 2320–6047
Vol. **1(8),** 1-7, September **(2013)** *Res. J. Mathematical & Statistical Sci.*

Reverse Order Laws for the con-s-k-EP Weighted Generalized Inverses

Krishnamoorthy S. and MuthugobaI B.K.N.
Ramanujan Research Centre, Department of Mathematics, Govt. Arts College (Autonomous), Kumbakonam, Tamilnadu, INDIA

Available online at: www.isca.in
Received 18th March 2013, revised 6th May 2013, accepted 17th July 2013

Abstract

If A is a con s –$\mathcal{k}-EP$ matrix, then the reverse order laws for the con-s-k-EP weighted generalized inverse of A (with respect to the given matrices M,N) is a matrix which satisfies $AA^{\dagger}A = A$, $A^{\dagger}A\ A^{\dagger} = A^{\dagger}$ and that $MAA^{\dagger}$ and that $AA^{\dagger}N$ are symmetric under certain conditions on M,N. It is shown that the weighted generalized inverse exists if and only if A N $A^TMA = A$, in which case the inverse is $N^TA^TM^T$. When M.N are identity matrices, this reduces to the well known result that the weighted generalized inverse of a con-s-k-EP matrix when it exists, must be A^T .

Keywords: con-s-k-EP matrix, generalized inverse, weighted generalized inverse, reverse order laws for the con-s-k-EP.

Introduction

Let $c_{n\times n}$ be the space of nxn complex matrices of order n. let C_n be the space of all complex n tuples. For A$\epsilon c_{n\times n}$.

Let $\bar{A}$, A^T, A^*, A^S , $\bar{A}^S$, $A^{\dagger}$, R(A), N(A) and $\rho(A)$ denote the conjugate, transpose, conjugate transpose, secondary transpose , conjugate secondary transpose, Moore-Penrose inverse range space, null space and rank of A respectively. A solution X of the equation AXA = A is called generalized inverse of A and is denoted by A^- . If A ϵ $c_{n\times n}$ then the unique solution of the equations A XA =A , XAX = X $[AX]^* =$ AX , $[XA]^* = XA$ [1] is called the Moore-Penrose inverse of A and is denoted by $A^{\dagger}$. A matrix A is called con-s-k-EP$_r$ if $\rho(A) = r$ and N(A) = N(A^TVK) (or) R(A)=R(KVA^T). Throughout this paper let "$\mathcal{k}$" be the fixed product of disjoint transposition in S_n = { 1,2,….n} and k be the associated permutation matrix.

Let us define the function $\boldsymbol{k}$ (x) = $\left(x_{k(1)}, x_{k(2)}, \ldots, x_{k(n)}\right)$.

A matrix A = (a_{ij}) ϵ $c_{n\times n}$ is s-k-symmetric if a_{ij}= $a_{n-k(j)+1,n-k(i)+1}$ for i, j = 1,2,…..n . A matrix A ϵ $c_{n\times n}$ is said to be Con-s-k-EP if it satisfies the condition $A_x = 0 <=> A^S\ \mathcal{k}\ (x) = 0$ or equivalently N (A) =N(A^TVK). In addition to that A is con-s-k-EP $<=> KVA$ is con-EP or AVK is con-EP and A is con-s-k-EP$<=> A^T$ is con-s-k-EP$_r$ moreover A is said to be con-s-k-EP$_r$ if A is con-s-k-EP and of rank r. For further properties of con-s-k-EP matrices one may refer "Con-s-k-EP matrices" by Krishnamoorthy, S., Gunasekaran, K. and Muthugobal, B.K.N., [2].

Definition 1.1: Let $A \in C_{n\times n}$ and let $M, N \in C_{n\times n}$, be two positive definite matrices. The unique matrix X which satisfies
i $AXA = A$ ii $XAX = X$ iii $(MAX)^T = MAX$ iv $(NXA)^T = NXA$
is called the weighted Moore-Penrose inverse of A and is denoted by $A^{\dagger}_{M,N}$ obviously for M=N=I_n the weighted Moore-Penrose inverse of A is the Moore-Penrose inverse of A.

For $A \in C_{n\times n}$, the sets of least-squares weighted generalized inverse of A({1,3M}-inverse of A), minimum-norm weighted generalized inverse of A({1,4N}-inverse of A) {1,2,3M}- inverse of A and {1,2,4N} - inverses of A, respectively are given by

$$A\{1,3M\} = \left\{X : AXA = A, (MAX)^T = MAX\right\},\quad A\{1,4N\} = \left\{X : AXA = A, (NXA)^T = NXA\right\}$$

$$A\{1,2,3M\} = \left\{X : AXA = A,\ XAX = X, (MAX)^T = MAX\right\},\quad A\{1,2,4N\} = \left\{X : AXA = A,\ XAX = X, (NXA)^T = NXA\right\}$$

For more results concerning generalized inverse reported by many researchers.[3, 4, 5]

The reverse order law for the Moore-Penrose inverse seems first to have been studied by Greville[6] in 1960. Since then there have been quite a number of papers on this subject see[7,8]**).**

In the paper "On reverse order laws for the weighted generalized inverse" studied by Zheng, B[9], for the first time the authors presented necessary and sufficient conditions for several types of reverse order laws for the weighted generalized inverse to hold. In this paper we after new necessary and sufficient conditions for the reverse order laws for the weighted generalized inverses of Con-s-k-EP matrix. The significance of our results lies in the fact that the conditions given in this paper, especially for the {1,2,3M} and {1,2,4N} - reverse order laws, are purely algebraic while the conditions given in "On reverse order laws for the weighted generalized inverse,". Zheng, B[9] are mostly rank conditions. We present necessary and sufficient conditions for the following inclusions.

$B\{1,3N\}A\{1,3M\}\subseteq(AB)\{1,3M\}$, $B\{1,4K\}A\{1,4N\}\subseteq(AB)\{1,4K\}$

$B\{1,2,3N\}A\{1,2,3M\}\subseteq(AB)\{1,2,3M\}$, $B\{1,2,4K\}A\{1,2,4N\}\subseteq(AB)\{1,2,4K\}$

These inclusions are also valid for Con-s-k-EP matrices. M, N and K are three positive definite matrices of order n×n and L respectively. Also, we consider the reverse order law for the weighted {1,3,4}-inverse. We give necessary and sufficient conditions for $B\{1,3N,4L\}A\{1,3M,4N\}\subseteq(AB)\{1,3M,4L\}$ and $(AB)\{1,3N,4L\}\subseteq B\{1,3N,4L\}A\{1,3M,4N\}$ in the case when M,N,K are positive definite matrices of order n×n.

Reverse order laws for the weighted {1,3}, {1,4}, {1,2,3} and {1,2,4} -inverse of Con-s-k-EP matrices: Let $A,B\in C_{n\times n}$ be Con-s-k-EP matrices and let $M\in C_{n\times n}$ $N\in C_{n\times n}$, and $L\in C_{n\times n}$ be three positive definite matrices. In this section, we give necessary and sufficient conditions for the following inclusion to hold:

$$B\{1,3N\}A\{1,3M\}\subseteq(AB)\{1,3M\} \quad \text{. . . (2.1)}$$

$$B\{1,4L\}A\{1,4N\}\subseteq(AB)\{1,4N\} \quad \text{. . . (2.2)}$$

$$B\{1,2,3N\}A\{1,2,3M\}\subseteq(AB)\{1,2,3M\} \quad \text{. . . (2.3)}$$

$$B\{1,2,4L\}A\{1,2,4N\}\subseteq(AB)\{1,2,4N\} \quad \text{. . . (2.4)}$$

The results from this section generalize those of paper by Xiong, Z [10]and Cvetkovic-Ilic [11] to the case of weighted generalized inverses. First, we will state the characterization of the sets A{1,3M} and A{1,4N} given in the reference paper "On reverse order laws for the weighted generalized inverse" [9].

Lemma 2.1 [14]: Let $A\in C_{n\times n}$ and $M,N\in C_{n\times n}$ are positive definite matrices. For $G\in C_{n\times n}$, we have,

i. $G\in A\{1,3M\}\Leftrightarrow A^TMAG=A^TM$, ii. $G\in A\{1,4N\}\Leftrightarrow GAN^{-1}A^T=N^{-1}A^T$

Obviously, we can conclude that $A\{1,3M\}=\left\{A^{\dagger}_{M,I_n}+\left(I_n-A^{\dagger}_{M,I_n}A\right)Y:Y\in C_{n\times n}\right\}$ and $A\{1,4N\}=\left\{A^{\dagger}_{I_m,N}+Z\left(I_m-AA^{\dagger}_{I_m,N}\right):Z\in C_{n\times n}\right\}$

Now, we will give a similar characterization of the sets A{1,2,3M} and A{1,2,4N}.

Theorem 2.1: Let $A\in C_{n\times n}$ be Con-s-k-EP matrix, $M,N\in C_{n\times n}$ be positive definite matrices. For $G\in C_{n\times n}$, we have,

i. $G\in A\{1,2,3M\}\Leftrightarrow A^TMAG=A^TMKVA^TKVG=A^TM$ and $GKVA^TVKA^{\dagger}_{M,I_n}=G$

ii. $G\in A\{1,2,4N\}\Leftrightarrow GAN^{-1}A^T=GKVA^TVKN^{-1}A^T=N^{-1}A^T$ and $A^{\dagger}_{I_m,N}KVA^TVKG=G$

Proof: i If $G\in A\{1,2,3M\}$, then $A^TMKVA^TVKG=A^T\left[MKVA^TVKG\right]^T=A^TM$

$$GKVA^TVKA^{\dagger}_{M,I_n}=GM^{-1}\left(MKVA^TVKG\right)M^{-1}\left(MKVA^TVKA^{\dagger}_{M,I_n}\right)=GM^{-1}\left(MKVA^TVKG\right)^TM^{-1}\left(MKVA^TVKA^{\dagger}_{M,I_n}\right)^T=G$$

If we suppose that $A^TMAG=A^TMKVA^TVKG=A^TM$ and $GKVA^TVKA^{\dagger}_{M,I_n}=G$ $GKVA^TVKA^{\dagger}_{M,I_n}=G$

we have that, $AGA = M^{-1}\left(MKVA^TVKA^{\dagger}_{M,I_n}\right)^T AGA = M^{-1}\left(MKVA^TVKA^{\dagger}_{M,I_n}\right)^T A = A$

Also, $GAG = GM^{-1}\left(A^{\dagger}_{M,I_n}\right)^T A^T MKVA^TVKG = GM^{-1}\left(MKVA^TVKA^{\dagger}_{M,I_n}\right)^T = G$

And $\left(MKVA^TVKG\right)^T = G^T KVAVKM = G^T A^T MKVA^TVKG = MAG$

The statement (ii) can be proved in a similar way. Throughout the paper, we will use the following lemma.

Lemma 2.2 [3]: Let, $A \in C_{n\times n}$, $M \in C_{m\times m}$ and $N \in C_{n\times n}$, where M and N are positive definite. Then, $A^{\dagger}_{M,N} = N^{-1/2}\left(M^{1/2}AN^{-1/2}\right)^{\dagger} M^{1/2}$.

In the next theorem we present new necessary and sufficient conditions (3 and 4) for (2.1) to hold.

Theorem 2.2: Let, $A \in C_{n\times n}$, $B \in C_{n\times n}$ be Con-s-k-EP matrix. If $M, N \in C_{n\times n}$ are positive definite matrices, then the following conditions are equivalent:

(1^0) $B\{1,3N\}A\{1,3M\} \subseteq (AB)\{1,3M\}$

(2^0) $BB^{\dagger}_{N,I_k}N^{-1}A^TMKVA^TVKKVB^TVK = \left(N^{-1}A^TMKVA^TB^TVK\right)$

(3^0) $B^{\dagger}_{N,I_k}A^{\dagger}_{M,N} \in (AB)\{1,3M\}$

(4^0) $B^{\dagger}_{N,I_k}A^{\dagger}_{M,N} \in (AB)\{1,2,3M\}$

Proof: From Corollary 2.2 [9], we have that (1^0) is equivalent with (2^0). Now, Let us prove that (3^0) is equivalent to (1^0) and that (4^0) is equivalent to (1^0).

Let $\tilde{A} = M^{1/2}KVA^TVKN^{-1/2}$ and $\tilde{B} = N^{1/2}KVB^TVK$. For $X \in C_{n\times n}$ and $Y \in C_{n\times n}$ put $\tilde{X} = N^{1/2}XM^{-1/2}$ and $\tilde{Y} = YN^{-1/2}$. It is easy to see that the following equivalences hold:

$X \in A\{1,3M\} \Leftrightarrow \tilde{X} = \tilde{A}\{1,3\}$, $Y \in B\{1,3N\} \Leftrightarrow \tilde{Y} = \tilde{B}\{1,3\}$, $YX \in AB\{1,3N\} \Leftrightarrow \tilde{Y}\tilde{X} = \left(\tilde{A}\tilde{B}\right)\{1,3\}$

Obviously, $B\{1,3N\}\ A\{1,3M\} \subseteq (AB)\{1,3M\} \Leftrightarrow \tilde{B}\{1,3\}\ \tilde{A}\{1,3\} \subseteq \tilde{A}\tilde{B}\{1,3\}$.

So (1^0) is equivalent to $\tilde{B}\{1,3\}\ \tilde{A}\{1,3\} \subseteq \left(\tilde{A}\tilde{B}\right)\{1,3\}$ . . . (2.5)

Now, by **Theorem 3.1** [11], we get that **(2.5)** is equivalent to $\tilde{B}\tilde{B}^{\dagger}\tilde{A}^T\tilde{A}\tilde{B} = \tilde{A}^T\tilde{A}\tilde{B}$, which is by **Lemma 2.2** equivalent to $KVB^TVKB^{\dagger}_{N,I_k}N^{-1}A^TMKVA^TB^TVK = N^{-1}A^TMKVA^TB^TVK$

Since $B^{\dagger}_{N,I_k}A^{\dagger}_{M,N} \in (AB)$ {1,3M} is equivalent to $\tilde{B}^{\dagger}\tilde{A}^{\dagger} \in \left(\tilde{A}\tilde{B}\right)\{1,2,3\}$, the proof follows by **Theorem 3.1** [2].
A similar result in the case of weighted {1,4}-inverses follows from **Theorem 2.2** by reversal of products.

Theorem 2.3: Let $A, B \in C_{n\times n}$ Con-s-k-EP matrix. If $N, L \in C_{n\times n}$ are positive definite matrices, then the following conditions are equivalent:

i. $KVA^TB^TVKL^{-1}B^TNA^{\dagger}_{I_m,N}KVA^TVK = KVA^TB^TVKK^{-1}B^TN$, ii. $B\{1,4L\}\ A\{1,4N\} \subseteq (AB)\{1,4K\}$

iii. $B^{\dagger}_{N,K}A^{\dagger}_{I_m,N} \in (AB)\{1,4L\}$, iv. $B^{\dagger}_{N,K}A^{\dagger}_{I_m,N} \in (AB)\{1,2,4K\}$

Proof: Let $\tilde{A} = KVA^T VKN^{-1/2}$ and $\tilde{B} = N^{1/2} KVB^T VKK^{-1/2}$. For $X \in C_{nxn}$ and $Y \in C_{nxn}$, let $\tilde{X} = N^{1/2} X$ and $\tilde{Y} = K^{1/2} YN^{-1/2}$. Now, the proof is similar to the one of **Theorem 2.2** and follows from **Theorem 3.2** [11].

Now, we will consider the reverse order law for the weighted {1,2,3}-inverses and weighted {1,2,4}-inverses. Remark that necessary and sufficient rank conditions for the reverse order law of {1,2,3} and {1,2,4}-inverses are given in paper "The reverse order law for {1,2,3}- and {1,2,4}- inverses of a two- matrix product" by Xiong, Z [10].

Theorem 2.4: Let $A, B \in C_{n\times n}$ If $M, N \in C_{n\times n}$ are positive definite matrices, then the following conditions are equivalent:

$\left(1^{00}\right)$ $B\{1,2,3N\} A\{1,2,3M\} \subseteq (AB)\{1,2,3M\}$

$\left(2^{00}\right)$ $KVB^T VKB^{\dagger}_{N,I_L} N^{-1} A^T MKVA^T B^T VK = N^{-1} A^T MKVA^T B^T VK$, and ($\left(KVA^T B^T VKB^{\dagger}_{N,I_K}\right)^{\dagger}_{M,N} KVABVKB^{\dagger}_{N,I_L} = KVB^T VKB^{\dagger}_{N,I_L}$ or $KVA^T B^T VK(AB)^{\dagger}_{M,I_K} = KVA^T VKA^{\dagger}_{M,N}$)

Proof: Let $\tilde{A} = M^{1/2} KVA^T VKN^{-1/2}$ and $\tilde{B} = N^{1/2} KVB^T VK$. For $X \in C_{nxn}$, $Y \in C_{nxn}$ and $Z \in C_{nxn}$, put $\tilde{X} = N^{1/2} XM^{-1/2}$, $\tilde{Y} = YN^{-1/2}$ and $\tilde{Z} = ZM^{-1/2}$. Then we have that $\tilde{A}\tilde{B}\tilde{Z}\tilde{A}\tilde{B} = \tilde{A}\tilde{B}$ if an only if $KVA^T B^T VKZKVA^T B^T VK = KVA^T B^T VK$ and that $\tilde{Z}\tilde{A}\tilde{B}\tilde{Z} = \tilde{Z}$ if and only if $ZKVA^T B^T VK = Z$. Also, $\left(\tilde{A}\tilde{B}\tilde{Z}\right)^T = \tilde{A}\tilde{B}\tilde{Z}$ if and only if $\left(MKVA^T B^T VKZ\right)^T = MKVA^T B^T VKZ$.

Hence, $\tilde{Z} \in \left(\tilde{A}\tilde{B}\right)\{1,2,3\} \Leftrightarrow Z \in (AB)\{1,2,3M\}$

Similarly, we get that $X \in A\{1,2,3M\}$ if and only if $\tilde{X} \in A\{1,2,3\}$ and that $Y \in B\{1,2,3N\}$ if and only $\tilde{Y} \in A\{1,2,3\}$.
Using **Lemma 2.2** we can easily prove the following,

$$\left(\tilde{A}\tilde{B}\tilde{B}^{\dagger}\right)^{\dagger} \tilde{A}\tilde{B}\tilde{B}^{\dagger} = \tilde{B}\tilde{B}^{\dagger} \Leftrightarrow \left(KVA^T B^T VKB^{\dagger}_{N,I_L}\right)^{\dagger}_{M,N} KVA^T B^T VKB^{\dagger}_{N,I_L} = KVB^T B^{\dagger}_{N,I_K} VK$$

$$\left(\tilde{A}\tilde{B}\right)\left(\tilde{A}\tilde{B}\right)^{\dagger} = \tilde{A}\tilde{A}^{\dagger} \Leftrightarrow \left(KVA^T B^T VK\right)\left(KVA^T B^T VK\right)^{\dagger}_{M,I_K} = \left(KVA^T A^{T^T} VK\right)_{M,N}$$

$$\tilde{B}\tilde{B}^{\dagger} \tilde{A}^T \tilde{A}\tilde{B} = \tilde{A}^T \tilde{A}\tilde{B} \Leftrightarrow KVB^T B^{\dagger}_{N,I_k} VKN^{-1} A^T MKVA^T B^T VK = N^{-1} A^T MKVA^T B^T VK$$

Now, the proof follows from **Corollary 3.1** [11]. The case of weighted {1,2,4}- inverses is treated completely analogously, and the corresponding result follows by taking adjoints or by reversal of products.

Theorem 2.5: Let $A, B \in C_{n\times n}$ be Con-s-k-EP matrices. If $N, L \in C_{n\times n}$ are positive definite matrices then the following conditions are equivalent:

i. $B\{1,2,4K\} A\{1,2,4N\} \subseteq (AB)\{1,2,4N\}$,

ii. $KVA^T B^T VKL^{-1} B^T NA^{\dagger}_{I_M,N} KVA^T VK = KVA^T B^T VKL^{-1} B^T N$ and $\left(A^{\dagger}_{I_M,N} KVA^T B^T VK\right)\left(A^{\dagger}_{I_M,N} KVA^T B^T VK\right)^{\dagger}_{N,L} = A^{\dagger}_{I_M,N} KVA^T VK$ or $\left(KVA^T B^T VK\right)^{\dagger}_{I_m,L} \left(KVA^T B^T VK\right) = B^{\dagger}_{N,L} KVB^T VK$.

Reverse order law for weighted {1,3,4} inverses:
In this section we consider the reverse order law for the weighted {1,3,4}-inverses. We give necessary and sufficient condition for $B\{1,3N,4L\} A\{1,3M,4N\} \subseteq (AB)\{1,3M,4L\}$ and $(AB)\{1,3M,4L\} \subseteq B\{1,3N,4L\} A\{1,3M,4N\}$

In the case when M, N, L are positive definite matrices of appropriate sizes. Also we give a very short proof that $(AB)\{1,3M,4L\} \subseteq B\{1,3N,4L\}A\{1,3M,4N\}$ is actually equivalent to $(AB)\{1,3M,4L\} = B\{1,3N,4L\}A\{1,3M,4N\}$ we will begin with two auxiliary results.

Lemma 3.1

Let $A, B \in C_{n\times n}$ be Con-s-k-EP matrices. If $M, N \in C_{n\times n}$ are positive definite matrices, then the following statements are equivalent:

i. $B \in A\{1,3M,4N\}$

ii. $A^T MKVA^T B^T VK = A^T$ and $KVB^T A^T VKN^{-1}A^T = A^T$

iii. There exists $Y \in C_{n\times n}$ such that $KVB^T VK = A^\dagger_{M,N} + \left(I - A^\dagger_{M,N}KVA^T VK\right)Y\left(I - KVA^T VKA^\dagger_{M,N}\right)$

Proof: Let $\tilde{A} = M^{1/2}KVA^T VKN^{-1/2}$ and $\tilde{B} = N^{1/2}KVB^T VKM^{-1/2}$. It is easy to see that $B \in A\{1,3M,4N\} \Leftrightarrow \tilde{B} \in \tilde{A}\{1,3,4\}$ $\tilde{A}^T\tilde{A}\tilde{B} = \tilde{A}^T \Leftrightarrow A^T MAB = A^T$ and $\tilde{B}\tilde{A}\tilde{A}^T = \tilde{A}^T \Leftrightarrow KVB^T A^T VKN^{-1}A^T = A^T$.

Also, there exist $\tilde{Y} \in C_{n\times n}$ such that $\tilde{B} = \tilde{A}^\dagger + \left(I - \tilde{A}^\dagger\tilde{A}\right)Y\left(I - \tilde{A}\tilde{A}^\dagger\right)$ if and only if there exists $Y \in C_{n\times n}$ such that $B = A^\dagger_{M,N} + \left(I - A^\dagger_{M,N}A\right)Y\left(I - AA^\dagger_{M,N}\right)$.

Now, the proof follows from **Lemma 1.2** [12]: Hence, $A\{1,3M,4N\} = A^\dagger_{M,N} + \left(I - A^\dagger_{M,N}A\right)Y\left(I - AA^\dagger_{M,N}\right) : Y \in C_{n\times n}$

Lemma 3.2: Let $A, B \in C_{n\times n}$ be Con-s-k-EP. If $M, N, K \in C_{n\times n}$ are positive definite matrices then

i. $B^\dagger_{N,L}B\left(AB\right)^\dagger_{M,L} = \left(AB\right)^\dagger_{M,L}$ ii. $\left(AB\right)^\dagger_{M,L}AA^\dagger_{M,L} = \left(AB\right)^\dagger_{M,L}$

Proof: By easy computation, we can show that $B^\dagger_{N,L}B\left(AB\right)^\dagger_{M,L} = \left(AB\right)\{1,2\}$

Since, $MKVA^T B^T VKB^\dagger_{N,L}KVB^T VK\left(KVA^T B^T VK\right)^\dagger_{M,L} = MKVA^T B^T VK\left(KVA^T B^T VK\right)^\dagger_{M,L}$

is symmetric and $LB^\dagger_{N,L}KVB^T VK\left(KVA^T B^T VK\right)^\dagger_{M,L}\left(KVA^T B^T VK\right) = B^T\left(B^\dagger_{N,L}\right)^T\left(AB\right)^T L\left(KVA^T B^T VK\right)^\dagger_{M,L}\left(KVA^T B^T VK\right)$

$$= B^T\left(B^\dagger_{N,L}\right)^T\left(AB\right)^T\left(\left(KVA^T B^T VK\right)^\dagger_{M,L}\right)^T L = B^T A^T\left(\left(KVA^T B^T VK\right)^\dagger_{M,L}\right)^T L = L\left(KVA^T B^T VK\right)^\dagger_{M,L}\left(KVA^T B^T VK\right)$$

we have $B^\dagger_{N,L}KVB^T VK\left(KVA^T B^T VK\right)^\dagger_{M,L} = \left(KVA^T B^T VK\right)^\dagger_{M,L}$. The identity (ii) can be proved similarly.

Theorem 3.1: Let $A, B \in C_{n\times n}$ be Con-s-k-EP matrices. Let $M, N, L \in C_{n\times n}$ be positive definite matrices. The following conditions are equivalent:

i. $B\{1,3N,4L\}A\{1,3M,4N\} \subseteq \left(AB\right)\{1,3M,4L\}$, ii. $\left(AB\right)^\dagger_{M,L} = B^\dagger_{N,L}A^\dagger_{M,N}$

Proof: Let $\tilde{A} = M^{1/2}KVA^T VKN^{-1/2}$ and $\tilde{B} = N^{1/2}KVB^T VKL^{-1/2}$. For $X, Y \in C_{nxn}$ Set $\tilde{X} = N^{1/2}KVA^T VKM^{-1/2}$ and $\tilde{Y} = L^{1/2}YN^{-1/2}$.

It is easy to see that $X \in A\{1,3M,4N\} \Leftrightarrow \tilde{X} \in \tilde{A}\{1,3,4\}$, $Y \in B\{1,3N,4L\} \Leftrightarrow \tilde{Y} \in \tilde{B}\{1,3,4\}$,

$YX \in (AB)\{1,3M,4L\} \Leftrightarrow \tilde{Y}\tilde{X} \in (\tilde{A}\tilde{B})\{1,3,4\}$ and $B\{1,3N,4L\}\,A\{1,3M,4N\} \subseteq (AB)\{1,3M,4L\}$,

$\Leftrightarrow \tilde{B}\{1,3,4\}\tilde{A}\{1,3,4\} \subseteq (\tilde{A}\tilde{B})\{1,3,4\}$

We can easily prove the next equivalence. $(AB)^{\dagger}_{M,L} = B^{\dagger}_{N,L}\,A^{\dagger}_{M,N} \Leftrightarrow (\tilde{A}\tilde{B})^{\dagger} = \tilde{B}^{\dagger}\tilde{A}^{\dagger}$

Now, the proof follows by **Theorem 2.1** [12].

Remark that several conditions equivalent to (b) can be found in reference paper "Inverse order rule for weighted generalized inverse" [13].

Theorem 3.2: Let $A, B \in C_{n\times n}$ be Con-s-k-EP matrices. Let $M, N, L \in C_{n\times n}$ be positive definite matrices. The following conditions are equivalent:

i. $(AB)\{1,3M,4L\} \subseteq B\{1,3N,4L\}\ A\{1,3M,4N\}$, ii. $(AB)\{1,3M,4L\} = B\{1,3N,4L\}\ A\{1,3M,4N\}$

Proof: $(1) \Rightarrow (2)$: For every $Z, X, Y \in C_{n\times n}$ such that

$$\left(KVA^TB^TVK\right)^{\dagger}_{M,L} + \left(I - \left(KVA^TB^TVK\right)^{\dagger}_{M,L}\left(KVA^TB^TVK\right)\right) Z \left(I - \left(KVA^TB^TVK\right)\left(KVA^TB^TVK\right)^{\dagger}_{M,L}\right)$$

$$= \left[B^{\dagger}_{N,L} + \left(I - B^{\dagger}_{N,L}\,KVB^TVK\right)Y\left(I - KVB^TVKB^{\dagger}_{N,L}\right)\right]\ \left[A^{\dagger}_{M,N} + \left(I - A^{\dagger}_{M,N}\,KVA^TVK\right)Y\left(I - KVA^TVKA^{\dagger}_{M,N}\right)\right]$$

Multiplying the last equality by $B^{\dagger}_{N,L}KVB^TVK$ from the left and by $AA^{\dagger}_{M,N}$ from the right, by **Lemma 3.2** we have

$$\left(KVA^TB^TVK\right)^{\dagger}_{M,L} + \left(B^{\dagger}_{N,L}KVB^TVK - \left(KVA^TB^TVK\right)^{\dagger}_{M,L}\left(KVA^TB^TVK\right)\right)$$

$$Z\left(AA^{\dagger}_{M,N} - \left(KVA^TB^TVK\right)\left(KVA^TB^TVK\right)^{\dagger}_{M,L}\right) = B^{\dagger}_{N,L}A^{\dagger}_{M,N}$$

For Z=0 we get $\left(KVA^TB^TVK\right)^{\dagger}_{M,L} = B^{\dagger}_{N,L}\,A^{\dagger}_{M,N}$ which implies

$B\{1,3N,4L\}\,A\{1,3M,4N\} \subseteq (AB)\{1,3M,4L\}$

$(ii) \Rightarrow (i)$: This is obvious.

Theorem 3.3: Let $A, B \in C_{n\times n}$ be Con-s-k-EP matrices and let $M, N, L \in C_{nxn}$ be positive definite matrices. The following conditions are equivalent:

i. $(AB)\{1,3M,4K\} \subseteq B\{1,3N,4L\}\,A\{1,3M,4N\}$ ii. $(AB)\{1,3M,4L\} = B\{1,3N,4L\}\,A\{1,3M,4N\}$

iii. $(AB)^{\dagger}_{M,L}\ B^{\dagger}_{M,L}A^{\dagger}_{M,N}$ and $\left(B = A^{\dagger}_{M,L}AB\ or\ A = ABB^{\dagger}_{N,L}\right)$

Proof: Let $\tilde{A} = M^{1/2}KVA^TVKN^{-1/2}$ and $\tilde{B} = N^{1/2}KVB^TVKL^{-1/2}$. For $X \in C_{n\times n}$ and $Y \in C_{n\times n}$ let $\tilde{X} = N^{1/2}XM^{-1/2}$ and $\tilde{Y} = L^{1/2}YN^{-1/2}$. We have that

$X \in A\{1,3M,4N\} \Leftrightarrow \tilde{X} \in \tilde{A}\{1,3,4\}$

$Y \in B\{1,3N,4L\} \Leftrightarrow \tilde{Y} \in \tilde{B}\{1,3,4\}$

$YX \in AB\{1,3N,4L\} \Leftrightarrow \tilde{Y}\tilde{X} \in (\tilde{A}\tilde{B})\{1,3,4\}$

Now, the proof follows from **Theorem 2.3**. It is interesting to remark that using **Theorem 2.4** [12], we can conclude that $\left(AB\right)\{1,3M,4L\} \subseteq B\{1,3N,4L\}\ A\{1,3M,4N\}$ Can be true only in the case when m ≤ n.

Conclusion

The concept of conjugate secondary range k-hermitian matrices is a generalization of conjugate secondary k-hermitian matrices. In this thesis we characterize the conjugate secondary range k-hermitian matrices and deal with the results analogous to the results of conjugate range hermitian matrices.

Conjugate secondary range k-hermitian matrices have wide variety of application. One of the applications is to find the isomers from the chemical structure. Further these matrices can also be applied to result from the given genetic code matrix to a permutation genetic code matrix. From which one can get different amino acid sequences.

References

1. Rao C.R. and Mitra S.K., "Generalized Inverse of Matrices and Its Applications", *Wiley and Sons, New York*, **(1971)**

2. Krishnamoorthy S., Gunasekaran K. and Muthugobal B.K.N., Con-s-k- EP matrices", *Journal of Mathematical Sciences and Engineering Applications*, **5(1),** 353–364 **(2011)**

3. Ben-Israel A. and Greville T.N.E., Generalized Inverses: Theory and Applications," *2nd Edition, Springer, New York,* 2003.

4. Harte R.E., Invertibility and singularity, *Dekker* **(1988)**

5. Wang G., Wei, Y. and Qiao S., Generalized inverses: theory and computat ions, *Science Press*, **(2003)**

6. Greville T.N.E., Note on the generalized inverse of a matrix product, *SIAM Rev.* **8**, 518-521 **(1966)**

7. Xiong Z. and Qin Y., Mixed type reverse order laws for the generalized inverses of an operator product," *Arab. J. Sci. Eng.,* **36**, 475-486 **(2011)**

8. Zheng B. and Xiong Z., The reverse order law for {1,2,3}- and {1,2,4}- inverses of multiple matrix product," *Linear Multilinear Algebra,* **58** , 765-782 **(2010)**

9. Zheng B. and Xiong Z., On reverse order laws for the weighted generalized inverse, *Arab. J. Sci. Eng.,* **34,(2A)**, 195-203 **(2009)**

10. Xiong Z. and Zheng B., The reverse order law for {1,2,3} and {1,2,4}- inverses of a two- matrix product," *Appl. Math. Lett.,* **21** , 649-655 **(2008)**

11. Cvetkovic-Ilic D. and Harte R., Reverse order laws in C*-algebras, *Linear Algebra Appl.* **434**, 1388-1394 **(2011)**

12. Cvetkovic-Ilic D.S. and Pavlovic V., A comment on some recent results concerning the reverse order law for {1,3,4}-inverses, *App. Math. Comput.,* **217**, 105-109 **(2010)**

13. Sun W. and Wei Y., Inverse order rule for weighted generalized inverse, *SIAM. J. Matrix Anal. Appl.,* **19(3)**, 772-775 **(1998)**

14. Liu D. and Yang H., The reverse order law for {1,3,4}- inverse of the product of two matrices, *Appl. Math. Comput.,* **215 (12),** 4293- 4303 **(2010)**

15. Sun W. and Wei Y., Triple reverse- order law for weighted generalized inverses," *Appl. Math. Comput.,* **125(2-3)**, 221-229 **(2002)**