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Short Communication

A Common Fixed Point Theorem in Fuzzy Metric Space

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Abstract

In this paper we are changing the contraction condition which is more generalized. We are proving a common fixed point theorem by using a general contraction condition. Many authors prove fixed point theorems on complete fuzzy metric space using different contraction conditions.

Keywords: Fuzzy metric space, fixed point, R-weakly commuting mappings

Introduction

The concepts of fuzzy sets was initially investigate by Zadeh[1] which laid the foundation of fuzzy mathematics. Subsequently, it was developed extensively by many authors and used in various fields. Many Authors like Deng[2], Erceg[3], Kaleva and Seikkala[4], Kramosil and Michalek[5], George and Veeramani[6], Chugh and Kumar[7], Mohd. Imdad and Javid Ali[8] have introduced the concept of fuzzy metric spaces in various ways and prove fuzzy version of some known fixed point theorems, Chugh and Kumar[7], Mohd. Imdad and Javid Ali[8].

Theorem A. Let A, B, S and T be mappingsfrom a completefuzzy metric space (X,M,*) into itself satisfying $A(X) \subset T(X)$, $B(X) \subset S(X)$ and $M(Ax,By,t) \geq r(M(Sx,Ty,t))$ for all $x,y \in X$, where $r:[0,1] \rightarrow [0,1]$ is a continuous function such that r(s)>s for each 0<s<1. Suppose that one of A,B,S and T is continuous, pairs (A,S) and (B,T) are R-weakly commuting on X. Then A,B,S and T have a unique common fixed point in X.

Note that Theorem A for a pair of R-weakly commuting mappings was proved by Vasuki[18] provided one of the mapping is continuous.

Mohd. Imdad and Javid Ali[8] proved the theorem B.
Theorem B. Let (X,M,*) be a complete fuzzy metric space and let A,B,S and T be self mappings of X satisfying the following conditions:
$A(X) \subset T(X)$ and $B(X) \subset S(X)$, $M(Ax,By,t) \geq \phi(\min\{M(Sx,Ty,t),M(Sx,Ax,t),M(By,Ty,t)\})$

For all $x,y \in X$, where $\phi:[0,1] \rightarrow [0,1]$ is a continuous function such that $\phi(s)>s$ for each 0<s<1. An d one of A(X) , B(X), S(X) and T(X) is a complete subspace of X, then A and S have a point of coincidence, B and T have a point of coincidence. Moreover, if the pairs (A,S) and (B,T) are coincidentally commuting, then A, B, S and T have a unique common fixed point. In this paper we change the contraction condition and got the same result.

Preliminaries

Definition 2.1 A Fuzzy set A in X is a function with domain X and values in [0,1].

Definition 2.2 A binary operation $*:[0,1]x[0,1] \rightarrow [0,1]$ is a continuous t-norm if {[0,1], $*$ } is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a,b,c,d \in [0,1]$.

Definition 2.3 The triplet (X,M,*) is a fuzzy metric space if Xis an arbitrary set,* is a continuous t-norm, M is a fuzzy set in $X^2x[0,\infty)$ satisfying the following conditions : M(x,y,0) = 0, M(x,y,t) = 1 for all t>0 iff x= y, $M(x,y,t) = M(y,x,t) \neq 0$ for $t \neq 0$, $M(x,y,t)*M(y,z,s)) \leq M(x,z,t+s)$, $M(x,y,.): [0, \infty) \rightarrow [0,1]$ is left continuous for all $x,y,z \in X$ and s,t>0.

Example 2.1 Every metric space induces a fuzzy metric space. Let (X,d) be a metric space. Define a*b=a.b and M(x,y,t) = $\frac{kt^n}{kt^n + md(x,y)}$ k,m,n,t∈ R^+. Then (X,M,*) is a fuzzy metric space. If we put k=m=n=1, we get M(x,Y,t) = $\frac{t}{t + d(x,y)}$
The fuzzy metric induced by a metric d is referred to as a standard fuzzy metric.

Definition 2.4 A sequence $\{x_n\}$ in a fuzzy metric space (X,M,*) is convergent to x∈ if $\lim_{n\to\infty}$ $M(x_n,x,t) = 1$ for each t>0

Definition 2.5 A sequence $\{x_n\}$ in a fuzzy metric space (X,M,*) is called Cauchy if $\lim_{n\to\infty}$ $M(x_{n+p},x_n,t) = 1$ for every t>0 and each p>0. (X,M,*) is complete if every Cauchy sequence in X converges in X.

Definition 2.6 A pair of self-mappings (f,g) of a fuzzy metricspace (X,M,*) is said to be weakly commuting (cf.[18]) if M(fgx,gfx,t)≥M(fx,gx,t).

Definition2.7. A pair of self mappings (f,g) of a fuzzy metric space (X,M,*) is said to be R-weakly commuting (cf.[18]) if there exist some R>0 such that M(fgx,gfx,t)≥M(fx,gx,t/R).

Example 2.2 Let X = R, the set of real numbers. Define a*b = a.b and M(x,y,t) = $\left(e^{\frac{x-y}{t}}\right)^{-1}$, for all x, y ∈ X and t>0 0, for all x,y∈ X and t = 0 Then (X,M,*) is a fuzzy metric Space.
If we define fx = 2x-1 and gx = x^2. Then we can show that-M(fgx,gfx,t) = $\left(e^{\frac{2(x-1)^2}{t}}\right)^{-1}$ = M(fx,gx,t/2)

Which shows that the pair (f,g) is R-weakly commuting for R= 2.

Results

Now, let (X,M,*) be a complete fuzzy metric space and let A,B,S and T be self mappings of X satisfying the following conditions:
A(X) ⊂T(X) and B(X) ⊂S(X) (3.1)
M(Ax ,By, t) ≥ M(Sx,Ty,t)*M(Sx, Ax, t)*M(By, Ty, t) (3.2)

For all x,y∈ X.
Then for any arbitrary x_0∈ X ,
By 3.1, we choose a point x_1 ∈ X such that $Ax_0 = Tx_1$ and for this point x_1, there exist a point x_2∈ X such that $Sx_2 = Bx_1$ and so on.
Continuing in this way, we can construct a sequence $\{y_n\}$ in X such that
- $y_{2n} = Tx_{2n+1} = Ax_{2n}$, $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for n = 0, 1, 2, (3.3)
Firstly we prove the following Lemma –

Lemma 3.1 Let A, B, S and T be self mappings of a fuzzy metric space (X,M,*) satisfying the condition 3.1 and 3.2. Then the sequence $\{y_n\}$ defined by 3.3 is a Cauchy sequence in X.

Proof For t>0 $M(y_{2n},y_{2n+1},t) = M(Ax_{2n},Bx_{2n+1},t) \geq M(Sx_{2n},Tx_{2n+1},t)*M(Sx_{2n},Ax_{2n},t)*M(Tx_{2n+1}Bx_{2n+1},t) = M(y_{2n-1},y_{2n},t)*M(y_{2n-1},y_{2n},t)*M(y_{2n},y_{2n+1},t) = M(y_{2n-1},y_{2n},t)$

Thus $\{M(y_{2n},y_{2n+1},t),\ n\geq 0\}$ is an increasing sequence of positive real numbers in [0,1] and therefore tends to a limit $l \leq 1$. We observe that l =1.

If not, l<1
Which on letting n ⟶ ∞ in 3.4 one gets
$k < l * l * l$ a contradiction yielding thereby l = 1
Therefore for every n∈ N, using analogous argument one can show that $\{M(y_{2n},y_{2n+1},t),\ n\geq 0\}$ is a sequence of positive real numbers in [0,1] which tends to a limit l =1.
Therefore $M(y_n,y_{n+1},t) > M(y_{n-1},y_n,t)$ and $\lim_{n\to\infty} M(y_n,y_{n+1},t) = 1$

Which shows that $\{y_n\}$ is a cauchy sequence in X. Now we prove our main result as follows:

Theorem 3.1 Let A,B,S and T be four self-mappings of a fuzzy metric space (X,M,*) satisfying the condition $M(Ax ,By, t) \geq M(Sx,Ty,t)*M(Sx, Ax, t)*M(By, Ty, t)$ for all $x,y \in X$. and t>0. If $A(X) \subset T(X)$ and $B(X) \subset S(X)$ and one of A(X), B(X),S(X) and T(X) is a complete subspace of X, then, i. A and S have a point of coincidence, ii. B and T have a point of coincidence.

Moreover, if the pairs (A, S) and (B,T) are coincidentally commuting, then A,B,S, and T have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Then following arguments of Fisher[7], one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X $y_{2n} = Tx_{2n+1} = Ax_{2n}$, $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for n = 0, 1, 2, Then due to Lemma 3.1 $\{y_n\}$ is a Cauchy sequence in X. Now suppose that S(X) is a complete subspace of X, the the subsequence $y_{2n+1} = Sx_{2n+2}$ must get a limit in S(X). Call it to be u and $v=S^{-1}u$. Then Sv=u. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of $\{y_{2n}\}$ being a subsequence of the convergent sequence $\{y_n\}$. On setting x=v and $y=x_{2n+1}$ in 3.2 one gets (fort>0), $M(Av,y_{2n+1},t) = M(Av,Bx_{2n+1},t), \geq M(Sv,Tx_{2n+1},t)*M(Sv, Av, t)*M(Bx_{2n+1}, Tx_{2n+1}, t), \geq M (u,y_{2n},t)*M(Av,u,t)*M(y_{2n+1},y_{2n},t)$

Which on letting $\lim_{n\to\infty}$ reduces to $M(Av,u,t) \geq M(Av,u,t)$ a contradiction. Therefore Av = u = Sv, which shows that the pair(A,S) has a point of coincidence. As $A(X) \subset T(X)$, Av = u implies that $u \in T(X)$. Let $w \in T^{-1}u$, Then Tw = u. Now using (3.2) again $M(y_{2n},Bw,t) = M(Ax_{2n},Bw,t), \geq M(Sx_{2n},Tw,t)*M(Sx_{2n}, Ax_{2n}, t)*M(Bw, Tw, t), \geq M(y_{2n-1},u,t)*M(y_{2n-1}, y_{2n}, t)*M(Bw, u, t)$

Which on letting $\lim_{n\to\infty}$ reduces to $M(u,Bw,t) \geq M(u,Bw,t)$

A contradiction. Therefore u = Bw. Thus we have shown u = Av = Bw= Tw . Thus Both pairs have point of coincidence. If one assumes T(X) t be complete, then an analogous argument establishes this claim.

The remaining two cases pertain essentially to the previous cases. Indeed if A(X) is complete, then $u \in A(X) \subset T(X)$ and if B(X) is complete, then $u \in B(X) \subset S(X)$. Thus (i) and (ii) are completely established.

Since the pairs (A,S) and (B,T) are coincidentally commuting at v and w respectively, then Au = A(Sv) = S(Av) = Su and Bu = B(Tw) = T(Bw) = Tu.

If $Au \neq u$, then for t>0, $M(Au,u,t) = M(Au,Bw,t), \geq M(Su,,Tw,t)*M(Su, Au, t)*M(Bw, Tw, t), \geq M(Au,,Tw,t)*M(Au, Au, t)*M(Tw, Tw, t), \geq M(Au,u,t)*1*1, \geq M(Au,u,t)$ a contradiction.

Therefore Au= u. Similarly, one can show that Bu = u. Thus u is a common fixed point of A,B,S and T. The uniqueness of a common fixed point follows easily. Also u remains the unique common fixed point of both pairs separately. This completes the proof.

Conclusion

The condition which we set in our result is more generalize and helping to prove some common fixed point theorems in different fuzzy metric spaces.

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