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Review Paper

A Review of Literature relating to Optimum Chemical Balance Weighing Designs

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Abstract

The weighing problem originated in a casual illustration furnished by Yates. This illustration later led to a precise formulation of the weighing problem by Hotelling. Over the years the problem has attained a distinctive growth, has branched out in different directions, and has acquired meanwhile the status of a problem in the design of experiments. Many statisticians were thoroughly studied the problem of construction of weighing designs. The weighing problem originally considered by Yates and Hotelling, is concerned with finding the weights of 'v' objects in 'n' weighing operations. In the latter developments, attention has been in the direction of obtaining "optimum weighing designs" i.e. the design in which each of estimated weights attains the minimum. The optimality has been determined by means of "efficiency". A good quality of work has been done on the problem of determining optimal designs in terms of the A-, D- and E-optimality criteria. In recent years there has been very rapid development in this area of experimental design. This paper presents a review of the available literature on optimum chemical balance weighing design and its construction.

Keywords: Weighing design, chemical balance weighing design, optimum chemical balance weighing design, Type I Criteria, D-optimality, A-optimality, E-optimality.

Introduction

Using a balance to measure the weight of an object or to compare the weight of two objects is called "Weighing", which has been undertaken for thousands of years. Every human being on our planet is affected by weights and measures in some way or other. From the moment we are born and all through our daily lives, weighing and measuring are an important and often vital part of our existence. Our bodies, the food we eat and all the products we use as an essential part of modern living have all been weighed and measured at some stage in their development. Weights and measures are definitely one of man's greatest and most important inventions, ranking alongside the wheel in the development of civilization. Commerce would not have progressed beyond the barter system without the invention of a system of weights and measures. There are three elements to the weighing story and each evolved over the 6,000 years of its history; first, we have the use and development of weights; then the different weighing machines and apparatus; and finally the introduction of weights and measures to control commercial transactions.

History of Weighing Designs

Study of weighing problem originated in a casual illustration furnished by Yates[1]. The precise formulation of such problems is to be found in Hotelling[2]. Hotelling and Yates[1] have shown that the individual weights may be determined more accurately by weighing the objects in combinations rather than weighing each one separately. Over the years the problem has attained a distinctive growth, has branched out in different directions, and has acquired meanwhile the status of a problem in the design of experiments. The problem has also become associated with the name of Hadamard and has given noticeable momentum to research in the extension of the Hadmard determinant problem. The experimental designs are applicable to a broad class of problems of measurement of similar objects. The chemical balance problem (in which objects may be placed in either of the two pans of the balance) is almost completely solved by means of designs constructed from Hadamard matrices.

Origin of the Problem

Yates[1] showed that if several light objects such as seeds are weighed in groups rather than individually as customary and next the weights of the individual objects are estimated by the method of least squares, then the precision of the estimates increases quite considerably. In the scheme suggested by Yates the objects are always placed on the same pan in a chemical balance or on a single pan in a spring balance.

Yates considered the problem: A chemist has seven light objects (a, b, c, d, e, f, g) to weigh, and the scale also required a zero correction, so that eight weighings are necessary. The standard error of each weighing is denoted by σ, the variance therefore by σ^2. Since the weight assigned to each object by customary techniques is the difference between the readings of the scale when carrying that object and when empty, the variance of the assign weight is $2\sigma^2$, and its standard error is $\sigma\sqrt{2}$. The improved technique suggested by Yates consists of weighing all seven objects together and also weighing them in groups of three so chosen that each object is weighed four times altogether, twice with any other object and twice without it. Calling the reading from the scale $y_1, y_2, \ldots, y_8$, we then have eight equations for determining the unknown weights

$$\begin{aligned}
a + b + c + d + e + f + g &= y_1 \\
a + b + c &= y_2 \\
a + d + e &= y_3 \\
a + f + g &= y_4 \\
b + d + f &= y_5 \\
b + e + g &= y_6 \\
c + d + g &= y_7 \\
c + e + f &= y_8
\end{aligned}$$

Any particular weight is found adding together the four equations containing it, subtracting the other four, and dividing by 4. Thus, estimate of object *a* is given by $\hat{a} = (y_1 + y_2 + y_3 + y_4 - y_5 - y_6 - y_7 - y_8)/4$ (1)

For the first terms in the expression for a each coefficient is ¼ so the variance of "a" is $\sigma^2/2$, which is only one-fourth that for the direct method. The standard error has been halved. If a degree of accuracy is required calling for repetition a certain number of times of the weighings by the direct method, then only one-fourth as many weighing are needed by Yates' method to procure the same accuracy in the average. This was the advantage of Yates' method over the direct method.

Improvement suggested by Hotelling

Hotelling[2] suggested that a further improvement can be possible in the Yates' method, if Yates' procedure were modified by placing in the other pan of the scale those objects not included in the weighing and thus using two pan chemical balance. Calling the readings $z_1, z_2, \ldots, z_8$, we can write the scheme of weighing operations (interchanging *c* and *d* of Yates scheme).

$$\begin{aligned}
a + b + c + d + e + f + g &= z_1 \\
a + b + c - d - e - f - g &= z_2 \\
a - b - c + d + e - f - g &= z_3 \\
a - b - c - d - e + f + g &= z_4 \\
-a + b - c + d - e + f - g &= z_5 \\
-a + b - c - d + e - f + g &= z_6 \\
-a - b + c + d - e - f + g &= z_7 \\
-a - b + c - d + e + f - g &= z_8
\end{aligned}$$

From these equations, $\hat{a} = (z_1 + z_2 + z_3 + z_4 - z_5 - z_6 - z_7 - z_8)/8$ (2)

A similar expression is obtained for each of the other unknowns. The variance of each unknown by this method is $\sigma^2/8$. This shows that precision of the estimates of the weight of the object increases further. The standard error is half that by Yates' method or quarter of its value by the direct method of weighing each object separately. The number of repetition required to procure a particular standard error in the mean is one-sixteenth that by the direct method.

Following the two procedures of weighings i.e. single pan and two pan weighing given by Yates and Hotelling respectively a number of authors provided the methods of construction and analysis of such designs, together with investigation of the precision of such design.

Definitions

Weighing Design: Weighing designs consists of n groupings of the p objects and suppose we want to determine the individual weights of p objects. We can fit the results into the general linear model as y = Xw+e, (3)

where y is an $n \times 1$ random column vector of the observed weights, $X = (x_{ij}),\ i = 1,2....,\ n,\ j = 1,2....\ p$, is an $n\times$p matrix of known elements with

$$x_{ij} = \begin{cases} +1 & \text{if the jth object is placed in the left pan in the } i^{th} \text{ weighing,} \\ -1 & \text{if the jth object is placed in the right pan in the } i^{th} \text{ weighing} \\ 0 & \text{if the jth object is not weighted in the } i^{th} \text{ weighing} \end{cases}$$

w is the p$\times$1 column vector representing the unknown weights of objects and e is an $n \times 1$ random column vector of error such that
E(e) = 0 and E(ee') = $\sigma^2 I_n$

where 0_n is the $n \times 1$ column vector with zero elements everywhere, I_n is the $n\times n$ identity matrix, "E" stands for the expectation and e' is used for transpose of e. E is the vector of the error component in the different observations and
$E \sim N(0, \sigma^2 I_N)$.

The normal equations estimating w are of the form $X'X\hat{w} = X'y$, (4)

Where $\hat{w}$ is the vector of the weights estimated by the least squares method.

A chemical balance weighing design is said to be singular or nonsingular, depending on whether the matrix X'X is singular or nonsingular, respectively. It is obvious that the matrix X'X is nonsingular if and only if the matrix X is of full column rank (= p). Now, if X is of full rank, that is, when X'X = S is nonsingular, the least squares estimate of w is given by $\hat{w} = (X'X)^{-1}X'y$ and the variance - covariance matrix of $\hat{w}$ is $Var(\hat{w}) = \sigma^2 (X'X)^{-1}$ (5)

Hotelling[2] has shown that the minimum attainable variance for each of the estimated weights in this case is σ^2/n and proved the theorem that each of the variance of the estimated weights attains the minimum if and only if $X'X=nI_p$.

Variance limit of estimated weights

Let **X** be an $n \times p$ matrix of rank *p* of a chemical balance weighing design and let m_j be the number of times in which j^{th} object is weighed, j=1,2,…., p (i.e. the m_j be the number of elements equal to -1and 1 in j^{th} column of matrix X). Then Ceranka et al.[3] proved the following theorems

Theorem: For any nonsingular chemical balance weighing design given by matrix, the variance of $\hat{w}_j$ of a particular such that $1 \leq j \leq p$ cannot be less than σ^2/m, where m=max $\{m_1, m_2, \ldots, m_p\}$.

Theorem: For any n x p matrix X, of a nonsingular chemical balance weighing design, in which maximum number of elements equal to -1 and 1 in columns is equal to m, each of the variances of the estimated weights attains the minimum if and only if
$X'X = mI_p$ (6)

Chemical Balance Weighing Design: When the objects are placed on two pans in a chemical balance, we shall call the weighings two pan weighing and the design is known as two pan design or chemical balance weighing design. In chemical balance the objects can be placed either on one pan or on both pans for each weighing. If in a weighing design, suppose we are given *p* objects weighed in *n* weighing operations, the elements of design matrix X = $\{x_{ij}\}$ takes the values as

x_{ij} = +1 if the j^{th} object is placed in the left pan in the i^{th} weighing,
= -1 if the j^{th} object is placed in the right pan in the i^{th} weighing,
= 0 if the j^{th} object is not weighted in the i^{th} weighing.

The n^{th} order matrix X = {x_{ij}} is known as the design matrix of the chemical balance. If n weighing operations are to determine the weights of p=n objects, the minimum variance that each estimated weight might have is σ^2/n.

Optimum Chemical Balanced Weighing Design: A nonsingular chemical bala- nce weighing design is said to be optimal for the estimating individual weights of objects if the variances of their estimators attain the lower bound given by,

$$Var(\hat{w}) = \frac{\sigma^2}{m},\ j = 1,2,\ldots\ldots\ p \qquad (7)$$

In other words, For any n x p matrix X of rank p, of a nonsingular chemical balance weighing design, in which maximum number of elements equal to -1 and 1 in columns is equal to m, then each of the variances of the estimated weights attains the minimum if and only if $X'X = mI_p$ (8)

In other words, an optimum design is given by X satisfying (8). In particular case when m=n we have the theorem given by Hotelling[2].

Type I criteria: Type I criteria i.e. Information-based criteria are related to the information matrix X'X for the design. This matrix is important because it is proportional to the inverse of the variance-covariance matrix for the least-squares estimates of the linear parameters of the model. These criteria can be divided into two classes according to the number of parameters used; the first class uses all parameters of the model. In this class possible criterion to consider are G–, D–, A–, E– and I–optimality criteria. Here we discuss only D–, A– and E–optimality criteria.

D–optimality: D–optimality is the most important and popular design criterion, introduced by wald[4], put the emphasis on the quality of the parameter estimates. D-optimality criterion is also known as the determinant criterion. The aim of D-optimality is essentially a parameter estimation criterion. This is the most extensively studied of all the design criteria. D–optimality is defined as

$$\max_{x_i, i=1,\ldots,n} |X'X| \equiv \min_{x_i, i=1,\ldots,n} |(X'X)^{-1}| \qquad (9)$$

which means maximizing the determinant of the information matrix, or equivalently, minimizing the determinant of the inverse of the information matrix.

A–optimality: A–optimality criterion showed the employed criterion of optimality which is the one that involves the use of Fisher's information matrix. An algebraic approach for constructing A–optimal design under generalized linear models was presented by Yang[5]. A–optimality is defined as

$$\min_{x_i, i=1,\ldots,n} trace(X'X)^{-1.} \qquad (10)$$

or equivalently, minimizing the average variance of the parameter estimates.

E–optimality: E–optimality introduced by Ehrenfeld but the Computations of E–optimal polynomial regression designs were introduced by Heiligers[6]. A method for computing E–optimal designs for a broad class of two parameter models was presented by Dette and Haines[7]. E–optimality is defined as

$$\max\ \lambda_{\min}(X'X) = \min\ \lambda_{\max}(X'X)^{-1} \qquad (11)$$

The procedure here builds on finding the design which maximizes the minimum eigen value of X'X or equivalently, minimize the maximum eigen value of $(X'X)^{-1}$. The aim of E-optimality is to minimize the maximum variance of all possible normalized linear combinations of parameter estimates.

Related Work

In the field of Weighing Designs enormous work has been done by many Statisticians. Study of weighing problem originated in a casual illustration furnished by Yates[1]. The precise formulation of such problems is to be found in Hotelling[2]. Hotelling and Yates have shown that the individual weights may be determined more accurately by weighing the objects in combinations rather than weighing each one separately. Hotelling[2] showed that the minimum attainable variance for each of the estimated weights for a chemical balance weighing design is $\sigma^2\ /n$. He showed that each of the variance of the estimated weights attained the lower hound if and only if $X' X = nI_p$. A design satisfying this condition is called an optimum chemical balance weighing design. In this case, several methods of construction optimum chemical balance weighing designs are available in the literature. Prominent among them are the works of Mood[8], Dey[9-10], Benerjee[11-13], Kishen[14], Raghavrao[15-17] and others.

Mood[8] proposed some solutions of the weighing problems projected by Hotelling[2]. The experimental designs are applicable to a broad class of problems of measurement of similar objects. The chemical balance problem (in which objects may be placed in either of the two pans of the balance) is almost completely solved by means of designs constructed from Hadamard matrices. He provided designs both for a balance which has a bias and for one which has no bias. He found that when p objects were weighed in $n \geq p$ weighings, the variances of the estimates of the weights were of the order of σ^2/n in the chemical balance case (σ^2 is the variance of a single weighing).

A good deal of work has been done on the problem of determining optimal designs in various classes and subclasses of *D(N,n)*. A detailed account of weighing designs can be obtained from Banerjee[11-13] and Raghavarao[15-17]. Some problems connected with the optimality of chemical balance weighing designs were considered by and Banerjee. Raghavarao has provided a fairly complete account of the basic results available in this area. Banerjee[11-13] has introduced the subject matter in general terms to research workers in applied sciences. He has shown that the arrangements afforded by a balanced incomplete block design can be used as an efficient chemical balance design. Such designs suffer from one drawback viz., there were only a few number of degrees of freedom left for the estimation of error-variance σ^2. To overcome this difficulty, Dey[9-10] has been suggested that the whole design may be repeated a certain number of times to get an estimate of the error variance. He made an attempt to give an alternative design where there was no necessity of such repetition. He also showed that these designs give a lesser variance of the estimated weights than the repeated design.

Raghavarao[15-17] and Bhaskarao[18] have studied the problem in terms of the A-, D-, and E-optimality criteria, but their results were applicable only within the subclasses of designs of D (N,n) whose information matrices can be written in the form

$$aI_n + bJ_{nm} \tag{12}$$

where a and b are real numbers, I_n is the n x n identity matrix and J_{nm} is the m x n matrix of ones. Kiefer[19] and Cheng[20] have shown that designs whose information matrices can be written in the forms NI_n and $(N-1)I_n + J_{nn}$ were optimal over all designs in D(N,n) with respect to very large classes of criteria. However, as noted by Cheng[20], optimal designs in D(N, n) and D'(N, n) do not always have information matrices of the form (12). Indeed, the results of Ehlich[21] and Payne[22] showed that when N= 2 (mod 4), then a design $d \in D'(N,n)$ whose information matrix has the form

$$M_d = \begin{pmatrix} (n-2)I_{\bar{n}_1} + 2J_{\bar{n}_1\bar{n}_1} & 0 \\ 0 & (n-2)I_{\bar{n}_2} + 2J_{\bar{n}_2\bar{n}_2} \end{pmatrix} \tag{13}$$

was D-optimal in D'(N,n) where $\bar{n}_1 = [n/2]$, $\bar{n}_2 = n - \bar{n}_1$, and [w] denotes the greatest integer not exceeding $w \geq 0$. Work by Galil and Kiefer[23-24] also shown that optimal designs need not have information matrices of the form given in (12).

Further Jacroux et al.[25] considered the problem of optimally weighing n objects with N weighings on a chemical balance. Several previously known results were generalized by them. In particular, the designs shown by Ehlich and Payne to be D-optimal in various classes of weighing designs where N=2 (mod 4) were shown to be optimal with respect to any optimality criterion of Type I as defined in Cheng. Several results on the E-optimality of weighing designs were also given. They mainly generalized the results given by Ehlich and Payne. In particular, they shown that a design in D'(N, n) where N=2 (mod 4) whose information matrix has the form (13) was uniquely optimal with respect to any Type I criterion and that such designs were uniquely D-optimal over all designs in D(N,n). They also extended the result of Raghavarao[17]. They have shown that certain designs which can be obtained from S_N matrices as defined in Raghavarao were E-optimal in various classes D(N, n).

Cheng et al.[26] developed a technique for finding optimum designs for weighing n objects in N weighings (N => n) on a chemical balance. Certain designs were shown to be optimal with respect to a large class of criteria (including the A- and D-criteria) for sufficiently large $N \equiv 2$ or 3 (mod 4). For small N, the result allows the elimination of a large number of competitors, and those that remain can be checked by a computer.

Several methods of construction of optimum chemical balance weighing designs without any restrictions on the number of objects placed on the either pan are available in the literature. Dey[9-10], Saha[27] , Kageyama and Saha[28] and others have shown how optimum chemical balance weighing designs can be constructed from the incidence matrices of balanced incomplete block designs for p = v objects. Various aspects of chemical balance weighing designs were studied by Shah and Sinha[29]. They have presented various theoretical aspects of optimality studies in the set-up of traditional experimental designs. Kageyama and Saha[30] have constructed optimum chemical balance weighing designs for $p = v + 1$ objects in $n = 4(r - \lambda)$ weighings from incidence matrices of balanced incomplete block designs for v treatments. In the same case, Ceranka and Katulska[31-33] have studied another method of construction. They gave the necessary and sufficient conditions under which a chemical balance weighing design for $v + 1$ objects was optimal. Also certain new construction methods of these optimum designs by utilizing the incidence matrices of BIB designs for v treatments were given. They studied some other methods of the construction of the design matrix X for a chemical balance weighing design problem (p = v + 1 objects) using the incidence matrices of some BIB designs for v treatments, which gave new optimum chemical balance weighing designs. They also studied the problem of estimating the individual weights of objects with minimum variances by using a weighing design with non-homogeneity of the variances of errors in the model. They proposed the necessary and sufficient conditions for optimum biased spring balance weighing designs with non-homogeneity of the variances of errors and for optimum chemical balance weighing designs with non-homogeneity of the variances of errors and the relations between these designs were investigated. They also found the new optimum weighing designs.

Some results of construction chemical balance weighing designs under the restriction on the number of objects placed on the either pan were given by Swamy[34], Ceranka et al.[35] and Ceranka and Katulska[36]. Ceranaka and Katulska[36] have shown relations between parameters of chemical balance weighing designs in situation, when matrix X of chemical balance weighing design was based on the incidence matrices of balanced incomplete block designs and on balanced bipartite block designs, respectively. Ceranka and Katulska[32] have constructed optimum chemical balance weighing designs from two incidence matrices of balanced incomplete block designs. Several methods of constructing matrix X are available in the literature Ambrozy and Ceranka.

In this area of chemical balance weighing designs enormous work has been done by Ceranka et al.[37-53]. Ceranka et al.[37] provided the way to deal with the problem of estimating individual weights of objects, using a chemical balance weighing design under the restriction on the number in which each object is weighed. A lower bound for the variance of each of the estimated weights from this chemical balance weighing design was obtained and a necessary and sufficient condition for this lower bound to be attained was given. The incidence matrix of ternary balanced block design was used to construct optimum chemical balance weighing design under the restriction on the number in which each object was weighed.

Ceranka et al.[38] studied the problem of estimating individual weights of objects, using a chemical balance weighing design under the restriction on the number of times in which each object was weighed. A lower bound for the variance of each of the estimated weights from this chemical balance weighing design was obtained and a necessary and sufficient condition for this lower bound to be attained was given. The incidence matrices of balanced bipartite block designs were used to construct the design matrix of chemical balance weighing designs under the restriction on the number in which each object was weighed.

Ceranka et al.[39]discussed the problem of estimating individual weights of objects using a chemical balance weighing design under the restriction on multiplicity of each object weighing. There were given the conditions under which the existence of an optimum chemical balance weighing design for $p = v$ objects implies the existence of an optimum chemical balance weighing design for $p = v + 1$ objects. A new method of constructing the optimum chemical balance weighing design for $p = v + 1$ objects was proposed. The construction was based on the incidence matrices of balanced bipartite block designs for v treatments.

Ceranka et al.[40] studied the problem of estimating individual weights of objects, using a chemical balance weighing design under the restriction on the number in which each object was weighted. A lower bound for the variance of each of the estimated weights from this chemical balance weighing design was obtained and a necessary and sufficient condition for this lower bound to be attained was given. The incidence matrices of balanced incomplete block designs and balanced bipartite block design were used to construct the design matrix X of optimum chemical balance weighing design under the restriction on the number in which each object is weighted.

Assuming that in each weighing operation not all objects are included, Ceranka et al.[41] studied the problem of estimating individual weights of objects in chemical balance weighing design. All variances of estimated weights were equal and they attained the lower bound. They proposed necessary and sufficient condition under which this lower bound was attained by variances of each of the estimated weights from this chemical balance weighing design. For the construction of the design matrix X of optimal chemical balance weighing design they used the incidence matrices of balanced bipartite block designs and ternary balanced block design.

The incidence matrices of ternary balanced block designs for v treatments have been used by Ceranka et al.[42-44] to construct chemical balance weighing designs for $p = v$ and $v+1$ objects with uncorrelated estimators of weights. Ceranka et al.[42] have shown that the chemical balance weighing design with uncorrelated estimators of weights be constructed from the incidence matrices of ternary balanced block designs and then they constructed chemical balance weighing designs with uncorrelated estimators of weights for $p = v + 1$ objects from incidence matrices of ternary balanced block designs for v treatments. Conditions under which the existence of a chemical balance weighing designs with uncorrelated estimators of weights for v objects implies the existence of the design with the same restrictions for $v + 1$ objects were given. The existence of a chemical balance weighing design with uncorrelated estimators of weights for $v + 1$ objects implies the existence of the design the same restrictions for $p < v + 1$ objects. Ceranka et al.[43] studied the problem of estimation of the weights of p objects in n weighings using chemical balance weighing design under the restriction $p_1 + p_2 < p$ where p_1 and p_2 represent the number of objects placed on the right and left pan, respectively. The incidence matrices of two ternary balanced block designs for v treatments were used to construct chemical balance weighing designs of $p=v + 1$ objects. The conditions for uncorrelated estimates of unknown weights were proposed. They studied two methods of construction of the design matrix X for a chemical balance weighing design for $p = v + 1$ objects under the restriction on the number of objects placed on either pan. These methods were based on two incidence matrices of ternary balanced block designs for v treatments.

Ceranka et al.[44-45] studied the estimation problem of individual weights of objects using a chemical balance weighing design under the restriction on the number times in which each object was weighed. Conditions under which the existence of an optimum chemical balance weighing design for $p = v$ objects implies the existence of an optimum chemical balance weighing design for $p = v + 1$ objects were given. The existence of an optimum chemical balance weighing design for $p = v + 1$ objects implies the existence of an optimum chemical balance weighing design for each $p < v + 1$. The new construction method for optimum chemical balance weighing design for $p = v + 1$ objects was given using the incidence matrices of ternary balanced block designs for v treatments.

For the case when the errors are correlated with equal variances, the conditions for determining the existence of the optimum chemical balance weighing design were considered in Ceranka et al.[46]. They proposed the lower bound of variance of each of the estimators and the construction methods of the optimal design. In the case of $G = I_n$, Wong et al.[47] proposed some construction methods of the A-optimal chemical balance weighing designs and also gave the lower bound for $\mathrm{tr}(XX)^{-1}$. Ceranka et al.[48] studied the estimation problem of individual measurements (weights) of objects in a model of chemical balance weighing design with diagonal variance - covariance matrix of errors under the restriction $k_1 + k_2 < p$, where k_1 and k_2 represent the number of objects placed on the right and left pans, respectively. They want all variances of estimated measurements to be equal and attaining their lower bound. For this they gave a necessary and sufficient condition under which this lower bound was attained by the variance of each of the estimated measurements. They proposed some methods of construction of an optimum chemical balance weighing design under the restriction on the number of objects placed on either of the pans. To construct the design matrix X of the considered optimum chemical balance weighing design they utilized the incidence matrices of balanced bipartite weighing designs. For the case of G being a positive definite diagonal matrix of known elements, Ceranka et al.[49] gave the lower bound of $\mathrm{tr}(X' G^{-1} X)^{-1}$ and the necessary and sufficient condition for this lower bound to be attained. The problem studied by them concerns the estimation of individual weights of p objects according to the model of an A-optimal chemical balance weighing design with a positive definite diagonal variance matrix of errors under the restriction $p_1 + p_2 = q \leq p$, where p_1 and p_2 represent the numbers of objects placed on the left and on the right pan respectively, in each of the measurement operations. The lower bound of $\mathrm{tr}(X' G^{-1} X)^{-1}$ was obtained and the necessary and sufficient condition for this lower bound to be attained under the given restriction on the number of objects included in the particular measurement operation was given. To construct the A-optimal chemical balance weighing design a set of incidence matrices of the balanced bipartite weighing designs were used. Ceranka et al.[50] studied the estimation problem of individual weights of p objects using the design matrix X of the A-optimal chemical balance weighing design under the restriction $p_1 + p_2 = q \leq p$, where p_1 and p_2 represent the number of objects placed on the left pan and on the right pan, respectively, in each of the measurement operations. The lower bound of $\mathrm{tr}(X'X)^{-1}$ was attained and the necessary and sufficient conditions for this lower bound to be obtained was given by them. They have given new construction methods of the A-optimal chemical balance weighing designs based on incidence matrices of the balanced bipartite weighing designs and the ternary balanced block designs.

Assuming that errors are uncorrelated with different variances, Ceranka et al.[51] studied the estimation problem of individual weights of objects using a chemical balance weighing design under the restriction on the number of times in which each object was weighed. The necessary and sufficient condition under which the lower bound of variance of each of the estimated weights is attained was given. For a new construction method of the optimum chemical balance weighing design they used the incidence matrices of the balanced incomplete block designs and the ternary balanced block designs.

The problem of estimation of individual measurements (weights) of *p* objects using *n* measurement operations according to the model of the chemical weighing design was presented by Ceranka et al.[52]. Assuming that not all objects in each measurement were included, the optimality conditions and the construction methods of the design matrix X of the optimum chemical balance weighing design for $p = v + 1$ objects were given. The construction was based on the incidence matrices of the balanced incomplete block designs and the balanced bipartite weighing designs for *v* treatments.

The problem of the estimation of unknown weights of $p = v + 1$ objects in the model of the chemical balance weighing design under the assumption that the measurement errors are uncorrelated and they have different variances was considered by Ceranka et al.[53]. The existence conditions determining the optimum design were presented. Ceranka et al.[54] utilized the incidence matrices of two group divisible designs with the same association scheme and the design matrix of chemical balance weighing design to construct optimum chemical balance weighing design.

Applications of weighing designs

Besides being helpful in routine weighing operations, the weighing designs, either the balanced ones or the others are applicable. i. to determine the weights of light objects, ii. useful in chemistry, physics, biological, economic and other sciences, iii. to a great variety of problems of measurement, not only of weights, but of lengths, voltages and resistances, iv. to determine concentration of chemicals in solutions, v. particularly useful in biological and chemical laboratories engaged in routine chemical analysis, vi. some special instances of balanced weighing designs, like in calibration, vii. to any problem of measurement in which the measure of a combination is expressible as a linear combination of the separate measures with numerically equal coefficients, viii. in general to any situation with additive effects.

Conclusion

It is difficult to weight the light objects accurately by weighing balance when measured individually. For getting high precision of the estimates, weighing designs are used. So if several light objects are weighed in groups rather than individually, we will obtain the estimates of high precision. Weighing designs are useful in various problems experimental designs. In this study the research and literature review were organized according to the construction and subject.

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