



## Integral Equations To Non-Riemann Cases

Seal Babulal

Department of Statistics, Burdwan University, Burdwan, W.B., INDIA

Available online at: [www.isca.in](http://www.isca.in)

Received 31<sup>st</sup> July 2013, revised 3<sup>rd</sup> August 2013, accepted 10<sup>th</sup> August 2013

### Abstract

The linear or nonlinear equations are important in many cases of stochastic process viz, renewal equations, age dependent branching process etc. There is a standard theory of linear integral equations. Here the technique is generalised and modified under different set ups. Generalisations are considered in three different cases: i. Integration is Riemann-Stieltjes integration with respect to an integrator of bounded variation, ii. integration w.r.t. step function integrator and iii. integration on any probability space or measure space.

**Keywords:** Non-Riemann integral equation, Fredholm integral equation of second kind, Riemann-Stieltjes integrator, step function, bounded variation function, measure space.

### Introduction

Integral equations come in many stochastic processes<sup>1</sup>. As linear integral equations have been studied extensively by others, we have not discussed those things, but we have generalised some of the results where the set up of the equations are different<sup>2</sup>. In fact the author tried to solve many nonlinear integral equations analytically and then those techniques are attempted to generalise. Such works are important in stochastic processes for solving many integral equations<sup>3</sup>. Also nonlinear integral equations are discussed in literature<sup>4</sup>. Motivated from this, other set ups for example, type of integration and integrator have been attempted. Three different cases: i. Integration is Riemann-Stieltjes integration with respect to an integrator of bounded variation, ii. integration w.r.t. step function integrator and iii. integration on any probability space or measure space are described and investigated. For general type of functional equation one may look at Evans<sup>5</sup>. One may use some tools for applications from literature<sup>6</sup>. One important reference in stochastic processes is Ross<sup>7</sup>. Another from application point of view is Ross<sup>8</sup>.

The solutions and conditions in the above mentioned three cases are obtained. Let us consider the following equation as in (1)

$$\varphi(x) = F(x) + \lambda \int_a^b K(x, \xi) \varphi(\xi) dg(\xi) \quad (1.1)$$

where  $g(\cdot)$  is any function of bounded variation

**Definition 1.1** (Apostol) A function  $g(\cdot)$  is called a bounded variation on  $[a, b]$  if for any partition  $P: a = x_0 < x_1 < \dots < x_n = b$  we have

$$\sup_P \sum_{i=1}^n |g(x_i) - g(x_{i-1})| \text{ is bounded}^9.$$

$$\varphi(x) = F(x) + \lambda \int_a^b K(x, \xi) \varphi(\xi) dG(\xi) \text{ as in (2)} \quad (1.2)$$

where  $G(\cdot)$  is a step function and discontinuity at  $c_1, c_2, \dots, c_s \in (a, b)$

$$\varphi(x) = F(x) + \lambda \int_X K(x, \xi) \varphi(\xi) d\mu(\xi) \text{ as in (3)} \quad (1.3)$$

where  $\mu$  is a finite measure on the sample space  $(X, \Gamma)$ .

where  $K: X \times X \rightarrow \mathbf{R}$ ,  $F: X \rightarrow \mathbf{R}$

These are considered in sections 2, 3 and 4. Then the solutions are obtained in each case. Some examples are also worked out.

**Solution of Integral Equation in Riemann-Stieltjes Function**

Let us consider (1.1) in Riemann-Stieltjes sense and we replace the integral by approximation sum as

$$F(x) = \phi(x) - \lambda \sum_{\gamma=1}^{n-1} K(x, \xi_\gamma) \phi(\xi_\gamma) [g(\xi_{\gamma+1}) - g(\xi_\gamma)] \tag{2.1}$$

where  $\xi_1 < \xi_2 < \dots < \xi_n$  are partitioning points of  $[a, b]$

Hence we get

$$F(\xi_1) = \phi(\xi_1) - \lambda \sum_{\gamma=1}^{n-1} K(\xi_1, \xi_\gamma) \phi(\xi_\gamma) [g(\xi_{\gamma+1}) - g(\xi_\gamma)] = \{1 - \lambda K(\xi_1, \xi_1) [g(\xi_2) - g(\xi_1)]\} \phi(\xi_1) - \lambda \sum_{\gamma=2}^{n-1} K(\xi_1, \xi_\gamma) \phi(\xi_\gamma) [g(\xi_{\gamma+1}) - g(\xi_\gamma)]$$

$$\Rightarrow \{1 - \lambda K(\xi_1, \xi_1) [g(\xi_2) - g(\xi_1)]\} \phi(\xi_1) - \lambda K(\xi_1, \xi_2) [g(\xi_3) - g(\xi_2)] \phi(\xi_2) - \lambda K(\xi_1, \xi_3) [g(\xi_4) - g(\xi_3)] \phi(\xi_3) - \dots - \lambda K(\xi_1, \xi_{n-1}) [g(\xi_n) - g(\xi_{n-1})] \phi(\xi_{n-1}) = F(\xi_1)$$

Similar expressions at  $\xi_2, \xi_3, \dots, \xi_{n-1}$

Proceeding like this for others we have  $n$  similar equations. (2.2)

In order to have a unique solution the coefficient matrix must be non-singular i.e.,

$$D_n(\lambda) = \begin{bmatrix} 1 - \lambda k_{11} g_{21} & -\lambda k_{12} g_{32} & \dots & -\lambda k_{1n} g_{n+1,n} \\ -\lambda k_{21} g_{21} & 1 - \lambda k_{22} g_{32} & \dots & -\lambda k_{2n} g_{n+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda k_{n1} g_{21} & -\lambda k_{n2} g_{32} & \dots & 1 - \lambda k_{nn} g_{n+1,n} \end{bmatrix}$$

where  $g_{ij} = g(\xi_i) - g(\xi_j)$  &  $k_{ij} = K(\xi_i, \xi_j)$ , must be non-singular.

$$\text{Now } \det(D_n(\lambda)) = 1 - \lambda \sum_{\gamma} K_{\gamma\gamma} g_{\gamma+1,\gamma} + \frac{(-\lambda)^2}{2!} \sum_{\gamma} \begin{vmatrix} k_{uu} g_{u+1,u} & k_{uv} g_{v+1,v} \\ k_{uv} g_{u+1,u} & k_{vv} g_{v+1,v} \end{vmatrix} + \frac{(-\lambda)^2}{3!} \dots$$

Now taking limit as  $n \rightarrow \infty$  and finer partition we have  $\lim_{n \rightarrow \infty} \det(D_n(\lambda)) = D(\lambda)$

In order to have solution of (2.2) we must have  $D(\lambda) \neq 0$ . Then the solution becomes,

**Theorem 2.1.** The solution in the previous case is given by  $\phi(x) = F(x) + \lambda \int_a^b R(x, \xi, \lambda) F(\xi) dg(\xi)$

where  $R(x, \xi, \lambda) = \frac{D(x, \xi, \lambda)}{D(\lambda)}$  &  $D(x, \xi, \lambda) = B_0(x, \xi) + \sum_{i=1}^{\infty} (-1)^i \frac{\lambda^i}{i!} B_i(x, \xi)$  &  $B_i(x, \xi) = \int_a^b \dots \int_a^b K \begin{pmatrix} x & \xi_1 & \dots & \xi_i \\ \xi & \xi_1 & \dots & \xi_i \end{pmatrix} dg(\xi_1) \dots dg(\xi_i)$

$$D(\lambda) = 1 - \sum_{i=1}^{\infty} (-1)^i \frac{\lambda^i}{i!} \int_a^b \dots \int_a^b K \begin{pmatrix} \xi_1 & \dots & \xi_i \\ \xi_1 & \dots & \xi_i \end{pmatrix} dg(\xi_1) \dots dg(\xi_i)$$

$$\text{where } K \begin{pmatrix} x & \xi_1 & \dots & \xi_n \\ \xi & \xi_1 & \dots & \xi_n \end{pmatrix} = \begin{vmatrix} K(x, \xi) & K(x, \xi_1) & \dots & K(x, \xi_n) \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) & \dots & K(\xi_1, \xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_n, \xi) & K(\xi_n, \xi_1) & \dots & K(\xi_n, \xi_n) \end{vmatrix}$$

**Proof:** We take

$$R(x, \xi; \lambda) = K_1(x, \xi) + \lambda K_2(x, \xi) + \lambda^2 K_3(x, \xi) + \dots \text{ and so on}$$

$$= K(x, \xi) + \lambda \left[ \int_a^b K(x, \xi_1) R(\xi_1, \xi; \lambda) dg(\xi_1) \right]$$

Now let us write  $D(x, \xi; \lambda) = R(x, \xi; \lambda)D(\lambda)$

Then putting this into previous expression we get  $\frac{D(x, \xi; \lambda)}{D(\lambda)} = K(x, \xi) + \lambda \int K(x, \xi_1) \frac{D(\xi_1, \xi; \lambda)}{D(\lambda)} dg(\xi_1)$   
 $\Rightarrow D(x, \xi; \lambda) = D(\lambda)K(x, \xi) + \lambda \int K(x, \xi_1)D(\xi_1, \xi; \lambda)dg(\xi_1)$  (2.3)

As we know  $D(\lambda) = 1 + \sum_{i=1}^{\infty} (-1)^i \frac{\lambda^i}{i!} P_i$  &  $P_i = \int_a^b \dots \int_a^b K \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_i \\ \xi_1 & \xi_2 & \dots & \xi_i \end{pmatrix} dg(\xi_1) \dots dg(\xi_i)$

Putting  $\lambda = 0$  as it is valid range because  $D(0) = 1 \neq 0$   $D(x, \xi; 0) = K(x, \xi)$

By previous argument the solution will be of the form  $D(x, \xi; \lambda) = B_0(x, \xi; \lambda) + \sum_{i=1}^{\infty} (-1)^i \frac{\lambda^i}{i!} B_i(x, \xi)$  (2.4)

Putting (2.4) in (2.3) we get  $\left( 1 + \sum_{i=1}^{\infty} (-1)^i \frac{\lambda^i}{i!} P_i \right) K(x, \xi) + \lambda \int K(x, \xi_1) \left( B_0(\xi_1, \xi; \lambda) + \sum_{i=1}^{\infty} (-1)^i \frac{\lambda^i}{i!} B_i(\xi_1, \xi) \right) dg(\xi_1)$   
 $= B_0(x, \xi; \lambda) + \sum_{i=1}^{\infty} (-1)^i \frac{\lambda^i}{i!} B_i(x, \xi)$

Now successively collecting coefficient of  $\lambda^i$  we have  $K(x, \xi)P_i = B_i(x, \xi) - i \int K(x, \xi_1)B_{i-1}(\xi_1, \xi)dg(\xi_1)$

$\Rightarrow B_1(x, \xi) = K(x, \xi)P_1 - \int_a^b K(x, \xi_1)B_0(\xi_1, \xi)dg(\xi_1) = K(x, \xi) \int_a^b K(\xi_1, \xi_1)dg(\xi_1) - \int_a^b K(x, \xi_1)B_0(\xi_1, \xi)dg(\xi_1)$   
 $= \int_a^b \begin{vmatrix} K(x, \xi) & K(x, \xi_1) \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) \end{vmatrix} dg(\xi_1)$  similarly for other  $B_i(x, \xi)$ .

**Solution of integral equation with step function integrator**

Now we consider equation (1.2) i.e., the integral equation when the integrator function  $g(\cdot)$  is discontinuous at  $y_1, y_2, \dots, y_k$ . Then we know that the Riemann-Stieltzes integral becomes (Apostol, 1973)

$\int_a^b K(x, y)\phi(y)dg(y) = \sum K(k, y_i)\phi(y_i)\alpha_i$  (3.1)

where

$\alpha_i = \text{jump of } g(y) \text{ function at } y_i \text{ for } i = 1, 2, \dots, k$

$\Rightarrow \phi(x) = F(x) + \lambda \sum_{i=1}^k K(x, y_i)\phi(y_i)\alpha_i, \forall x$

Now putting  $y_1, y_2, \dots, y_k$  in both sides in the above we have  $k$  equations like (3.1).

Above system of equations can be rewritten as

$F(y_1) = (1 - \lambda K(y_1, y_1)\alpha_1)\phi(y_1) - \lambda K(y_1, y_2)\alpha_2\phi(y_2) - \dots - \lambda K(y_1, y_k)\alpha_k\phi(y_k)$   
 $F(y_2) = -\lambda K(y_2, y_2)\alpha_2\phi(y_2) + (1 - \lambda K(y_2, y_1)\alpha_1)\phi(y_1) - \dots - \lambda K(y_2, y_k)\alpha_k\phi(y_k)$   
 $F(y_k) = -\lambda K(y_k, y_1)\alpha_1\phi(y_1) - \dots - \lambda K(y_k, y_{k-1})\alpha_{k-1}\phi(y_{k-1}) + (1 - \lambda K(y_k, y_k)\alpha_k)\phi(y_k)$  (3.2)

Now in order to solve the above system of equations (3.2) we must have

$$D(\lambda) = \begin{bmatrix} 1 - \lambda K(y_1, y_1)\alpha_1 & -\lambda K(y_1, y_2)\alpha_2 & \cdots & -\lambda K(y_1, y_k)\alpha_k \\ -\lambda K(y_2, y_1)\alpha_1 & 1 - \lambda K(y_2, y_2)\alpha_2 & \cdots & -\lambda K(y_2, y_k)\alpha_k \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda K(y_k, y_1)\alpha_1 & -\lambda K(y_k, y_2)\alpha_2 & \cdots & 1 - \lambda K(y_k, y_k)\alpha_k \end{bmatrix}$$

must be non-singular.

$$\begin{aligned} \text{Now } \det(D(\lambda)) &= 1 - \lambda \sum_{i=1}^n K(y_i, y_i)\alpha_i + \frac{(-\lambda)^2}{2!} \sum_{i,j} \begin{vmatrix} K(y_i, y_i)\alpha_i & K(y_i, y_j)\alpha_j \\ K(y_j, y_i)\alpha_i & K(y_j, y_j)\alpha_j \end{vmatrix} + \dots \\ &= 1 - \lambda \sum_{i=1}^n K(y_i, y_i)\alpha_i + \frac{(-\lambda)^2}{2!} \sum_{i,j} \alpha_i \alpha_j \begin{vmatrix} K(y_i, y_i) & K(y_i, y_j) \\ K(y_j, y_i) & K(y_j, y_j) \end{vmatrix} + \dots \text{and so on} \end{aligned} \quad (3.3)$$

Let us take special case when  $g(\cdot)$  has only one discontinuity points at  $x = c$  then  $D(\lambda) = 1 - \lambda K(c, c)\alpha$  where  $\alpha$  is amount of jump of  $g$  at  $x = c$ .

$$\Rightarrow \lambda \neq \frac{1}{K(c, c)\alpha}$$

in order to have solution and the solution is given by

$$\phi(x) = F(x) + \int_a^b R(x, \xi, \lambda) F(\xi) d\xi$$

where

$$R(x, \xi, \lambda) = \frac{D(x, \xi; \lambda)}{D(\lambda)} \text{ and } D(x, \xi; \lambda) = K(x, \xi) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_a^b \cdots \int_a^b K \begin{pmatrix} x & \xi_1 & \cdots & \xi_n \\ \xi & \xi_1 & \cdots & \xi_n \end{pmatrix} dg(\xi_1) \cdots dg(\xi_n)$$

by theorem 2.1.

$$\text{In this case we must have } \alpha_1 = \alpha, \text{ and } \alpha_2 = \alpha_3 = \cdots = \alpha_n \Rightarrow \lambda \neq \frac{1}{\alpha K(c, c)}$$

In the following we shall workout an example

**Example 1.** Let us take  $K(x, \xi) = xe^\xi$ ,  $a = 0, b = 1$

$$\text{Then } K \begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} = \begin{vmatrix} K(x, \xi) & K(x, \xi_1) \\ K(\xi, \xi) & K(\xi_1, \xi_1) \end{vmatrix} = \begin{bmatrix} xe^\xi & x e^{\xi_1} \\ \xi e^\xi & \xi_1 e^{\xi_1} \end{bmatrix} = x \xi_1 e^{\xi + \xi_1} - x \xi e^{\xi + \xi_1} = 0 \Rightarrow \int_a^b \begin{vmatrix} K(x, \xi) & K(x, \xi_1) \\ K(\xi, \xi) & K(\xi_1, \xi_1) \end{vmatrix} d\xi_1 = 0$$

and other higher order determinants must be zero.

Now we see using theorem 3.1 that

$$B_0(x, \xi; \lambda) = K(x, \xi)$$

$$\Rightarrow D(x, \xi; \lambda) = K(x, \xi)$$

$$\Rightarrow R(x, \xi; \lambda) = \frac{K(x, \xi)}{D(\lambda)}$$

$$\Rightarrow D(x, \xi; \lambda) = K(x, \xi) = xe^\xi$$

$$\Rightarrow R(x, \xi; \lambda) = \frac{xe^{\xi}}{1 - \lambda K(c, c)\alpha}$$

For  $\lambda = 1$

$$R(x, \xi) = \frac{xe^{\xi}}{1 - K(c, c)\alpha} \text{ and the solution is given by}$$

$$\begin{aligned} \phi(x) &= F(x) + \int_0^1 \frac{xe^{\xi}}{1 - K(c, c)\alpha} F(\xi) dg(\xi) \\ &= F(x) + \frac{xe^{\xi}}{1 - K(c, c)\alpha} F(c)\alpha \text{ is a solution} \end{aligned} \tag{3.4}$$

**Compact form of solution:** Now let us take  $\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\phi(\xi)dg(\xi)$

where  $y_1, y_2, \dots, y_k$  are the points of discontinuity of  $g(\cdot)$ .

$$\text{So above becomes } \phi(x) = F(x) + \lambda \sum_{i=1}^k K(x, y_i)\phi(y_i)\alpha_i$$

Let us try with  $\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \dots$

$$\text{Then } \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \dots = F(x) + \lambda \sum_{i=1}^k K(x, y_i)[\phi_0(y_i) + \lambda\phi_1(y_i) + \lambda^2\phi_2(y_i) + \dots]\alpha_i$$

$$\Rightarrow \phi_0(x) = F(x)$$

$$\phi_1(x) = \sum_{i=1}^k K(x, y_i)\phi_0(y_i)\alpha_i = \sum_{i=1}^k K(x, y_i)F(y_i)\alpha_i$$

$$\phi_2(x) = \sum_{i=1}^k K(x, y_i)\phi_1(y_i)\alpha_i = \sum_{i=1}^k K(x, y_i) \left[ \sum_{j=1}^k K(y_i, y_j)F(y_j)\alpha_j \right] \alpha_i = \sum_{j=1}^k \left[ \sum_{i=1}^k K(x, y_i)K(y_i, y_j)\alpha_i \right] F(y_j)\alpha_j = \sum_{j=1}^k K_2(x, y_j)F(y_j)\alpha_j$$

$$\phi_n(x) = \sum_{i=1}^k K(x, y_i)\phi_{n-1}(y_i)\alpha_i = \sum_{j=1}^k K_n(x, y_j)F(y_j)\alpha_j$$

$$\text{where } K_2(x, y_j) = \sum_{i=1}^k K(x, y_i)K(y_i, y_j)\alpha_i \quad K_3(x, y_j) = \sum_{i=1}^k K(x, y_i)K_2(y_i, y_j)\alpha_i \quad K_n(x, y_j) = \sum_{i=1}^k K(x, y_i)K_{n-1}(y_i, y_j)\alpha_i$$

These are successive iterated kernels.

$$\text{Now consider the series } R(x, \xi; \lambda) = \sum_{i=0}^{\infty} \lambda^i K_{i+1}(x, \xi)$$

$$\text{Then solution becomes as in the following } \phi(x) = \sum_{i=0}^{\infty} \lambda^i \phi_i(x) = F(x) + \lambda \sum_{i=0}^{\infty} \lambda^i \phi_{i+1}(x) = F(x) + \lambda \sum_{i=0}^{\infty} \lambda^i \left( \sum_{j=1}^k K_{i+1}(x, y_j)F(y_j)\alpha_j \right)$$

$$= F(x) + \lambda \sum_{j=1}^k \left( \sum_{i=0}^{\infty} \lambda^i K_{i+1}(x, y_j) \right) F(y_j) \alpha_j = F(x) + \lambda \sum_{j=1}^k R(x, y_j; \lambda) F(y_j) \alpha_j \quad (3.5)$$

**Solution of integral equation in measure theoretic set up**

Now let us consider the last case: Here we partition the sample space into  $A_1, A_2, \dots, A_n$  s.t.

$$\phi(x) - \lambda \sum_{i=1}^{\infty} K(x, \xi_i) \phi(\xi_i) \mu(A_i) \cong F(x) \quad (4.1)$$

where  $\xi_i \in A_i$ .

The above expression approximation can be done upto any desired level because we know that for bounded functions and finite measure integrals are approximated in this way.

It is known that this can be done uniformly over  $x$  and the set on which function is defined, is a compact set and the  $\sigma$ -field is compatible with the topological structure<sup>10</sup>. Then we have the following system of equation

$$\begin{bmatrix} 1 - \lambda k_{11} \mu(A_1) & -\lambda k_{12} \mu(A_2) & \dots & -\lambda k_{1n} \mu(A_n) \\ -\lambda k_{21} \mu(A_1) & 1 - \lambda k_{22} \mu(A_2) & \dots & -\lambda k_{2n} \mu(A_n) \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda k_{n1} \mu(A_1) & -\lambda k_{n2} \mu(A_2) & \dots & 1 - \lambda k_{nn} \mu(A_n) \end{bmatrix} \begin{bmatrix} \phi(\xi_1) \\ \phi(\xi_2) \\ \vdots \\ \phi(\xi_n) \end{bmatrix} = \begin{bmatrix} F(\xi_1) \\ F(\xi_2) \\ \vdots \\ F(\xi_n) \end{bmatrix}$$

where  $k_{ij} = K(\xi_i, \xi_j)$

Then we must have the non-singular coefficient matrix

$$D_p(\lambda) = 1 - \lambda \sum_{\gamma} k_{\gamma\gamma} \mu(A_{\gamma}) + \frac{\lambda^2}{2!} \sum \begin{vmatrix} k_{rr} \mu(A_r) & k_{rs} \mu(A_s) \\ k_{sr} \mu(A_r) & k_{ss} \mu(A_s) \end{vmatrix} + \dots$$

Then taking finer and finer partitions, the limit value coincides to its integral. So we have

$$D(\lambda) = 1 - \lambda EK(\xi, \xi) + \frac{\lambda^2}{2!} E_{\mu} \begin{bmatrix} K(\xi_1, \xi_1) & K(\xi_1, \xi_2) \\ K(\xi_2, \xi_1) & K(\xi_2, \xi_2) \end{bmatrix} + \text{other terms}$$

Here second expression is w.r.t product measure.

So we have

**Theorem 4.1.** If  $X$  is a compact topological space and  $\mu$  is defined on its  $\sigma$ -field then the solution for the above integral equation is given by

$$\phi(x) = F(x) + \lambda \int_{\Omega} R(x, \xi, \lambda) F(\xi) d\mu(\xi)$$

where  $R(x, \xi, \lambda) = \frac{D(x, \xi, \lambda)}{D\lambda}$  and  $D(x, \xi, \lambda) = B_0(x, \xi) + \sum_{i=1}^{\infty} (-1)^i \frac{\lambda^i}{i!} B_i(x, \xi)$

and  $B_i(x, \xi) = \int_{\Omega} \dots \int_{\Omega} K \begin{pmatrix} x & \xi_1 & \dots & \xi_i \\ \xi & \xi_1 & \dots & \xi_i \end{pmatrix} d\mu(x) d\mu(\xi_1) \dots d\mu(\xi_i)$

$$D(\lambda) = 1 - \sum_{i=1}^{\infty} (-1)^i \frac{\lambda^i}{i!} \int_{\Omega} \dots \int_{\Omega} K \begin{pmatrix} \xi_1 & \dots & \xi_1 \\ \xi_1 & \dots & \xi_1 \end{pmatrix} d\mu(\xi_1) \dots d\mu(\xi_i)$$

$K(\cdot, \cdot)$  is a measurable, bounded function on the product  $\sigma$ -field.  $F(\cdot)$  is a given measurable function.

**Proof:** Holds as before, by partitioning the space into  $B_1, B_2, \dots, B_{k(x)}$  such that for given  $x$ , the integral can be approximated. Similarly for other  $x$  also. Moreover, by compactness  $\exists$  finite number of sets  $A_1, A_2, \dots, A_k$  such that  $\forall x$  the integral can be approximated by these sets and rest of the proof is as before.

## Conclusion

We have extended the idea of power series technique to linear equations but with different setup. For non-linear case there is scope for further study.

## References

1. Bhat U.N., Elements of Applied Stochastic Processes, 2nd Ed., Wiley, New York, (1984)
2. Volterra V., Lecons sur la theorie mathematiques de la lutte pour la vie, Paris, 214 (1931)
3. Hoel P.G., et al. Introduction to Stochastic Processes, Universal Book Stall, New Delhi, (1991)
4. Davis H.T., Introduction to Nonlinear Differential and Integral Equations, 1st ed., Dover Publications, Inc., New York, (1962)
5. Evans G.C., Some General Types of Functional Equations, *Fifth International Congress of Mathematicians*, **1(1)**, 387-396 (1912)
6. Jordan C., Calculus of Finite Differences, 3rd ed., New York: Chelsea, (1965)
7. Ross S.M., Introduction to Probability Models. 6th ed., New York: Academic Press, (1994)
8. Ross S.M., Applied Probability Models with Optimization Applications. Dover Publication, (1970)
9. Apostol T.M., Mathematical Analysis. Addison-Wesley Publishing Company, Massachusetts, (1973)
10. Simmons G.F., Introduction to Topology and Modern Analysis. McGraw-Hill, (1963)