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Inversion of an Integral Involving a Product of General Class of Polynomials and $\overline{H}$-function as Kernel

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Abstract

Solution for a certain convolution integral equation of Fredholm types whose kernel involves a product of general class of polynomials and $\overline{H}$ -function has been obtained. The main result is believed to be general and unified in nature. A number of results follow as special cases by specializing the parameters of the general class of polynomials and $\overline{H}$ -function.

Keywords: Convolution integral equation, general class of polynomials, $\overline{\mathrm{H}}$-function.

Introduction

Fredholm type integral equations with polynomial or special function kernels have been studied earlier by many research workers[1-4]. The main object of this paper is to evaluate an exact solution of the following convolution integral equation of Fredholm type

$$\int_0^\infty y^{-1} u\left(\frac{x}{y}\right) \phi(y) dy = g(x), \quad (x > 0) \tag{1}$$

where g is a prescribed function, ϕ is an unknown function to be determined and the kernel $u(x)$ is given by

$$u(x) = S_N^M[x] \overline{H}_{p,q}^{m,n}[x], \tag{2}$$

where $S_N^M[x]$ is the general class of polynomials introduced and studied by Srivastava[5] and is defined by means of the following series expansion:

$$S_N^M[x] = \sum_{R=0}^{[N/M]} \frac{(-N)_{M_R}}{R!} A_{N,R}\, x^R, \quad N = 0,1,\cdots, \tag{3}$$

where M is an arbitrary positive integer and the coefficients $A_{N,R}$ $(N,\ R \geq 0)$ are arbitrary constants real or complex and $\overline{H}_{p,q}^{m,n}[x]$ is the Inayat Hussain $\overline{H}$-function[6] and studied by Buchman and Srivastava[7], which can be written in terms of Mellin-Barnes type integral by

$$\overline{H}_{p,q}^{m,n}[x] = \overline{H}_{p,q}^{m,n}\left[x \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,n};(a_j,\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m};(b_j,\beta_j,B_j)_{m+1,q} \end{matrix} \right.\right] = \frac{1}{2\pi\omega}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\ \prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)} x^{\xi} d\xi\,, (z \neq o) \tag{4}$$

where $\mathcal{L}$ is a suitable contour. The details of this function can be found in[6, 7, 8].

Our method of solution of integral equation (1) with kernel $u(x)$ given by (2) would depend on the theory of Mellin transform defined by $\Phi(s) = \mathcal{M}\{\phi(x); s\} = \int_0^\infty x^{s-1}\phi(x) dx,$ (5)
provided that the integral exists.

Mellin Transform of u(x)

In order to solve the integral equation, we shall require the following result contained in
Lemma: Let $\mathrm{U(s)} = \mathcal{M}\{\mathrm{u(x); s}\}$ where u(x) is defined by (2), then

$$U(s) = \sum_{R=0}^{[N/M]} \frac{(-N)_{M_R}}{r!} A_{N,R}\, x^R \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j(R+s))\ \prod_{j=1}^{n}\{\Gamma(1-a_j-\alpha_j(R+s))\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j-\beta_j(R+s))\}^{B_j}\ \prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j(R+s))} \tag{6}$$

provided that $-\min_{1\le j\le m} Re\left(\frac{b_j}{\beta_j}\right) < Re(R+s) < \min_{1\le j\le n} Re\left(\frac{1-a_j}{\alpha_j}\right); R = 0,1,2,\ldots.[N/M]$

Proof of the lemma: Let

$$U(s) = \mathcal{M}\left\{S_N^M[x]\overline{H}_{p,q}^{m,n}[x]; s\right\} = \int_0^\infty x^{s-1} S_N^M[x]\overline{H}_{p,q}^{m,n}[x]\,dx. \tag{7}$$

Making use of the series expansion of the general class of polynomials defined by (3) and changing order of summation and integration, we get

$$U(s) = \sum_{R=0}^{[N/M]} \frac{(-N)_{MR}}{R!} A_{N,R}\, \mathcal{M}\left\{x^r \overline{H}_{p,q}^{m,n}[x]; s\right\}. \tag{8}$$

Further, apply the following known formulae[7] and Mellin Transform of $\overline{H}$ -function:

$$\mathcal{M}\{x^\rho \phi(x); s\} = \Phi(s+\rho) \tag{9}$$

And

$$\mathcal{M}\left\{\overline{H}_{p,q}^{m,n}[ax]\right\} = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\ \prod_{j=1}^{n}\{\Gamma(1-a_j-\alpha_j s)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j-\beta_j s)\}^{B_j}\ \prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)} a^{-s}, \tag{10}$$

provided that

$-\min_{1\le j\le m} Re\left(\frac{b_j}{\beta_j}\right) < Re(s) < \min_{1\le j\le n} Re\left(\frac{1-a_j}{\alpha_j}\right)$, we arrive at the desired result (6).

Solution of the Integral Equation

The solution of the convolution integral equation of Fredholm type is given in the following theorem:

Theorem: Let the Mellin transforms $\Phi(s), G(s)$ and $U(s) \ne 0$ of the functions $\phi(x), g(x)$ and $u(x)$ defined by (2) exist and are analytic in some infinite strip $\gamma < Re(s) < \delta$ of the complex s-plane. Also assume that for a fixed $c \in (\gamma, \delta)$, $u^*(x)$ be defined by

$$u^*(x) = \mathcal{M}^{-1}\{U^*(s); x\} = \frac{1}{2\pi\omega}\int_{c-\omega\infty}^{c+\omega\infty} x^{-s}\, U^*(s) ds \tag{11}$$

where,

$$U^*(s) = \left[\rho^k \frac{\Gamma\left(\frac{-s}{\rho}\right)}{\Gamma\left(\frac{-(s+\rho k)}{\rho}\right)} \sum_{R=0}^{[N/M]} \frac{(-N)_{MR}}{(R)!} A_{N,R} \bullet \frac{\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j(R+s+\rho k+\sigma)\right)\ \prod_{j=1}^{n}\{\Gamma(1-a_j-\alpha_j(R+s+\rho k+\sigma)).\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j-\beta_j(R+s+\rho k+\sigma)).\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j(R+s+\rho k+\sigma))}\right]^{-1}, \tag{12}$$

provided that the following sets of conditions hold

i. $|\arg x| < \frac{1}{2}\Omega\pi$, where Ω as defined in[6] , ii. $\rho \ne 0$ and k is a non-negative integer, iii. $-\min_{1\le j\le m} Re\left(\frac{b_j}{\beta_j}\right) < Re(R+s+\rho k+\sigma) < \min_{1\le j\le n} Re\left(\frac{1-a_j}{\alpha_j}\right)$ and iv. M is an arbitrary positive integer and the coefficients $A_{N,R}(N,R \ge 0)$ are arbitrary (real or complex) constants.

The integral equation (1) has its solution given by $\phi(x) = x^{-\rho k-\sigma}\int_0^\infty y^{-1}u^*\left(\frac{x}{y}\right)\left(y^{\rho+1}D_y\right)^k\{y^\sigma g(y)\}\,dy,$ (13)

provided that the integral exists.

Proof

Applying the convolution theorem for Mellin transforms[9], we find from (1), that $U(s)\Phi(s) = G(s)$ (14)

where $\Phi(s)$ and $G(s)$ are Mellin transforms of $\phi(x)$ and $g(x)$ respectively and $U(s)$ is given in (6). Replacing s by $s+\rho k+\sigma$ $(\rho \ne 0$ *and* k *is non-negative integer*$)$ in (14), we get

$$F(s+\rho k+\sigma) = U^*(s)\left\{\rho^k\left(-\frac{s+\rho k}{\rho}\right)_k\right\}G(s+\rho k+\sigma), \tag{15}$$

and making use of the formula given by Srivastava[4]:

$$\mathcal{M}\{(x^{h+1}D_x)^n f(x); s\} = h^n\left(-\frac{s+hn}{h}\right)_n F(s+hn), \ (h \ne 0, n \text{ a non-negative integer}), \tag{16}$$

we get $F(s+\rho k+\sigma) = U^*(s)[(y^{\rho+1}D_x)^k\{y^\rho\, g(y)\}\,;s]$ and applying (9), we find that (17)

$$\mathcal{M}\{x^{\rho k+\sigma}\phi(x);s\} = \mathcal{M}\left\{\int_0^\infty y^{-1}u^*\left(\frac{x}{y}\right)(y^{\rho+1}D_x)^k\{y^\rho\, g(y)\}\,;s\right\}. \tag{18}$$

Inverting both sides of (18) by using Mellin inversion theorem [7], we get the desired theorem.

Applications

i. If we set $M=1, A_{N,R} = \binom{N+\alpha}{N}\frac{(\alpha+\beta+N+1)_R}{(\alpha+1)_R}$, then the general class of polynomials $S_N^M[x]$ is reduced to Jacobi polynomial $P_N^{(\alpha,\beta)}(1-2x)$, and we get

Corollary 1: Under the hypothesis of Theorem, the integral equation $\int_0^\infty y^{-1}u_1\left(\frac{x}{y}\right)\phi(y)dy = g(x); (x>0),$ (19)

where the kernel $u_1(x) = P_N^{(\alpha,\beta)}(1-2x)\overline{H}_{p,q}^{m,n}[x]$ (20)
has its solution $\phi(x)$ given by

$$\phi(x) = x^{-\rho k-\sigma}\int_0^\infty y^{-1}u_1^*\left(\frac{x}{y}\right)\left(y^{\rho+1}D_y\right)^k\{y^\sigma\, g(y)\}\,dy \tag{21}$$

provided that the integral (21) exists and $u_1^*(x)$ is the Melline inverse transform of

$$U_1^*(s) = \left[\rho^k\frac{\Gamma\left(\frac{-s}{\rho}\right)}{\Gamma\left(\frac{-(s+\rho k)}{\rho}\right)}\sum_{R=0}^N\frac{(-N)_{MR}}{(R)!}\binom{N+\alpha}{N}\frac{(\alpha+\beta+N+1)_R}{(\alpha+1)_R}\frac{\prod_{j=1}^m\Gamma\left(b_j+\beta_j(r+s+\rho k+\sigma)\right)\prod_{j=1}^n\{\Gamma(1-a_j-\alpha_j(r+s+\rho k+\sigma)).\}^{A_j}}{\prod_{j=m+1}^q\{\Gamma(1-b_j-\beta_j(R+s+\rho k+\sigma)).\}^{B_j}\prod_{j=n+1}^p\Gamma(a_j+\alpha_j(R+s+\rho k+\sigma))}\right]^{-1} \tag{22}$$

provided that sets of conditions (i), (ii) and (iii) of the main Theorem are satisfied.

ii. On taking $A_j = B_j = 1$ the $\overline{H}$ -function reduces to Fox's H-function and we found that the solution of integral equation (1) in terms of Fox's H-function given in[3, p. 73].

iii. Further, letting $N=0$ and $A_{0,0}=1$, the general class of polynomials $S_N^M[x]$ is reduced to unity and we get a result given in [3].

Conclusion

It may be pointed out here that the general class of polynomials is very general in nature and it unifies and extends a number of classical orthogonal polynomials, such as Jacobi polynomials, the Laguerre polynomials, the Hermite polynomials, the Gagenbauer polynomials, the Legendre polynomials and several other classes of the generalized hypergeometric polynomials. Also the $\overline{H}$-function is a very general function and has for its particular cases a number of important special functions [6and7,8].

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