# Solving Linear Tri-level Programming Problem using Approaches Based on Line Search and an Approximate Algorithm 

HosseiniEghbal ${ }^{1}$ and Nakhai KamalabadiIsa ${ }^{2}$<br>${ }^{1}$ Candidate at Payamenur University of Tehran, Department of Mathematics, Tehran<br>${ }^{2}$ Industrial Engineering at University of Kurdistan, Sanandaj, IRAN

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#### Abstract

In the recent years, the bi-level and tri-level programming problems (TLPP) are interested by many researchers and TLPP is known as an appropriate tool to solve the real problems in several areas such as economic, traffic, finance, management, and so on. Also, it has been proven that the general TLPP is an NP-hard problem. The literature shows a few attempts for solving using TLPP. In this paper, we attempt to develop two effective approaches, one based on Taylor theorem and the other based on the hybrid algorithm by combining the penalty function and the line search algorithm for solving the linear TLPP. In these approaches, by using Karush-Kuhn-Tucker conditions, TLP Pischanged to non-smooth single problem, and then it is smoothed by proposed functions. Finally, the smoothed problem is solved using both of the proposed approaches. The presented approaches achieve an efficient and feasible solution in an appropriate time which has been evaluated by comparing to references and test problems.


Keywords: Linear bi-level programming problem, Linear tri-level programming problem, Karush-Kuhn-Tucker conditions, Taylor theorem, Line search method.

## Introduction

It has been proven that the multi-level programming problem, especially bi-level programming problem (BLPP), are NPHard problems ${ }^{1,2}$. However a few algorithms have been proposed to solve TLPP, several algorithms have been proposed to BLPP $^{3-9}$. These algorithms are divided into the following classes: global techniques, enumeration methods, transformation methods, meta- heuristic approaches, fuzzy methods, primal-dual interior methods. In the following, these techniques are shortly introduced.

Global techniques: All optimization methods can be divided into two distinctive classes: local and global algorithms. Local ones depend on initial point and characteristics such as continuity and differentiability of the objective function. On the other hand, global methods can achieve global optimal solution. These methods are independent of initial point as well as continuity and differentiability of the objective function ${ }^{10-13}$.

Enumeration methods: Branch and bound is an optimization algorithm that uses the basic enumeration. But in these methods we employ clever techniques for calculating upper bounds and lower bounds on the objective function by reducing the number of search steps.

These methods search in the all feasible vertex points which one of them is optimal solution.

Transformation methods: These methods are interested by some researchers for solving BLPP, so that they transform the follower problem by methods such as penalty functions, barrier functions, Lagrangian relaxation method or KKT conditions. In fact, these techniques convert the BLPP into a single problem and then it is solved by other methods ${ }^{15-18}$.

Meta heuristic approaches: Meta heuristic approaches are proposed by many researchers to solve complex combinatorial optimization. Whereas these methodsare too fast and known as suitable techniques for solving optimization problems, however, they can only propose a solution near to optimal. These approaches are generally appropriate to search global optimal solutions in very large space whenever convex or non-convex feasible domain is allowed. In these approaches, BLPP is changed to theone-level problem according to transformation methods and then meta-heuristic methods are utilized to find out the optimal solution ${ }^{19-26}$.

Fuzzy methods: Sometimes assigning crisp values to the variables, constraints, and objective functions are not appropriate. Therefore, in these cases, the fuzzy approach is an eligible tool to overcome their ambiguousness. In this category, membership functions can be leader, follower or both of objective functions also it can be define with constraints and variables. There are so many researchers using this method ${ }^{27-33}$.

Interior point methods: The interior point methods formulate many large linear programs to non-linear state and solve them with nonlinear algorithms. In these methods all iterates need to
satisfy the inequality constraints. The primal-dual method is a class of these methods which is the most efficient practical approach. In interior point methods can be strong competitors to the linear algorithm on bigproblems ${ }^{34,35}$.

The remainder of the pages are structured as follows: basic concepts of linear BLPP and TLPP are introduced in Section 2,. We provide a smooth method to BLPP and TLPP in Section 3. The presented algorithm is proposed in Section 4. Computational results are presented for our approaches in fifth Section. As result, the paper is finished in Section 6 by presenting the concluding remarks.

## The linear bi-leveland tri-level programming problems

In this section models of bi-level and tri-level programming problems are introduced. BLPP is used frequently by problems with decentralized planning structure. It is defined as ${ }^{36}$.
$\min _{x} f(x, y)=a^{T} x+b^{T} y$
s.t $\min _{y}^{x} g(x, y)=c^{T} x+d^{T} y$
$A x+B y \leq r$,
$\mathrm{x}, \mathrm{y} \geq 0$.

Wherea, $c \in R^{n_{1}} . b, d \in R^{n_{2}}, A \in R^{m \times n_{1}} . B \in R^{m \times n_{2}}, r \in R^{m}, x \in$ $\mathrm{R}^{\mathrm{n}_{1}}, \mathrm{y} \in \mathrm{R}^{\mathrm{n}_{2}}$ and $f(x, y)$ and $g(x, y)$ are the objective functions of the first level and second level.

In general, TLPP is not convex problem; therefore, there is no any global algorithm for solving it. This problem can be nonconvex even when all functions and constraints are bounded and continuous. Of course, the linear BLPP is convex and preserving this property is very important.

Because a tri-level programming problem describes the principle particulars of multi-level programming, the proposed algorithms will be developed for tri-level programming which can be extended for general multi-level programming problems which the number of levels is more than three. Hence, just trilevel programming is studied in this paper.

To describe a TLPP, a main model can be written as follows:
$\min _{\mathrm{x}} F_{1}(\mathrm{x}, \mathrm{y}, \mathrm{z})=a_{1} \mathrm{x}+b_{1} \mathrm{y}+c_{1} z$
$A_{1}^{\mathrm{x}} \mathrm{x}+B_{1} \mathrm{y}+C_{1} \mathrm{z} \leq r_{1}$,
s.t $\min _{\mathrm{y}} F_{2}(\mathrm{x}, \mathrm{y}, \mathrm{z})=a_{2} \mathrm{x}+b_{2} \mathrm{y}+c_{2} \mathrm{z}$
$A_{2} \mathrm{x}+B_{2} \mathrm{y}+C_{2} \mathrm{z} \leq r_{2}$,
s.t $\min _{\mathrm{z}} F_{3}(\mathrm{x}, \mathrm{y}, \mathrm{z})=a_{3} \mathrm{x}+b_{3} \mathrm{y}+c_{3} z$
$A_{3} \mathrm{x}+B_{3}^{\mathrm{z}} \mathrm{y}+C_{3} \mathrm{z} \leq r_{3}$,
$\mathrm{x}, \mathrm{y}, \mathrm{z} \geq 0$.
Where $\quad A_{i} \in \mathrm{R}^{\mathrm{q} \times \mathrm{k}}, B_{i} \in \mathrm{R}^{\mathrm{q} \times 1}, C_{i} \in \mathrm{R}^{\mathrm{q} \times \mathrm{p}}, r_{i} \in \mathrm{R}^{\mathrm{q}}, \mathrm{x} \in \mathrm{R}^{\mathrm{k}}, \mathrm{y} \in$ $\mathrm{R}^{\mathrm{l}}, \mathrm{z} \in \mathrm{R}^{\mathrm{p}}, a_{i} \in \mathrm{R}^{\mathrm{k}}, b_{i} \in \mathrm{R}^{\mathrm{l}}, c_{i} \in \mathrm{R}^{p}, \mathrm{i}=1,2,3$, and the $\mathrm{x}, \mathrm{y}, \mathrm{z}$ are
variables which called the first-level, second-level, and third-level variables respectively, and , $F_{1}(\mathrm{x}, \mathrm{y}, \mathrm{z}), F_{2}(\mathrm{x}, \mathrm{y}, \mathrm{z}), F_{3}(\mathrm{x}, \mathrm{y}, \mathrm{z})$, the first-level, second-level, and third-level objective functions, respectively.

Some definitions and notations are introduced in order togain an optimal solution for TLP problem based on the solution concept of bi-level programming [6],

Definition 2.1: The feasible region of the TLP problem when $\mathrm{i}=1,2,3$, is
$\mathrm{S}=\left\{(\mathrm{x}, \mathrm{y}, \mathrm{z}) \mid A_{i} \mathrm{x}+B_{i} \mathrm{y}+C_{i} \mathrm{z} \leq r_{i}, \mathrm{x}, \mathrm{y}, \mathrm{z} \geq 0.\right\}$
On the other hand, if x be fixed, the feasible region of the follower can be explained as

$$
\begin{equation*}
\mathrm{S}=\left\{(\mathrm{y}, \mathrm{z}) \mid B_{i} \mathrm{y}+C_{i} \mathrm{z} \leq r_{i}-A_{i} \mathrm{x}, \mathrm{y}, \mathrm{z} \geq 0\right\} \tag{4}
\end{equation*}
$$

Based on the above assumptions, the follower rational reaction set is
$P(x)=\{(y, z) \in \operatorname{argming}(x, y, z),(y, z) \in S(x)\}$.
Where the inducible region is as follows
$I R=\{(x, y, z) \in S,(y, z) \in P(x)\}$.
Finally, the tri-level programming problem can be written as $\min \{F(x, y, z) \mid(x, y, z) \in \operatorname{IR}\}$.

If there is a finite solution for the TLP problem, we define feasibility and optimality for the TLP problem as
$\mathrm{S}=\left\{(\mathrm{x}, \mathrm{y}, \mathrm{Z}) \mid A_{i} \mathrm{x}+B_{i} \mathrm{y}+C_{i} \mathrm{z} \leq r_{i}, \mathrm{x}, \mathrm{y}, \mathrm{z} \geq 0\right\}$.
Definition 2.2: Every point such as ( $\mathrm{x}, \mathrm{y}, \mathrm{z}$ ) is a feasible solution to tri-level problem if $(x, y, z) \in I R$

Definition 2.3: Every point such as $\left(x^{*}, y^{*}, z^{*}\right)$ is optimal solution if
$F\left(x^{*} \cdot y^{*}, z^{*}\right) \leq F(x, y, z) \forall(x, y, z) \in I R$.

## Smooth method for TLPP

Using KKT conditions for both of last levels in problem (2), the following problem is constructed:
$\min _{\mathrm{x}} F_{1}(\mathrm{x}, \mathrm{y}, \mathrm{z})=a_{1} \mathrm{x}+b_{1} \mathrm{y}+c_{1} \mathrm{z}$
s.t $A_{1} \mathrm{x}+B_{1} \mathrm{y}+C_{1} \mathrm{z}-r_{1} \leq 0$,
$A_{2} \mathrm{x}+B_{2} \mathrm{y}+C_{2} \mathrm{z}-r_{2} \leq 0$,
$A_{3} \mathrm{x}+B_{3} \mathrm{y}+C_{3} \mathrm{z}-r_{3} \leq 0$,
$\mu\left(A_{3} \mathrm{x}+B_{3} \mathrm{y}+C_{3} \mathrm{z}-r_{3}\right)=0$,
$\mu B_{3}=-b_{3}$,
$\beta\left(A_{2} \mathrm{x}+B_{2} \mathrm{y}+C_{2} \mathrm{z}-r_{2}\right)=0$,
$\beta C_{2}=-c_{2}$,
$\mathrm{x}, \mathrm{y}, \mathrm{z}, \mu, \beta \geq 0$.
Because problem (10) has a complementary constraint, it is not convex and it is not differentiable. In this paper a smooth method is proposed for smoothing complementary constraints in
problem (10). Using the following smooth method, problem (10) will be smoothed, and then we present two algorithms based on Taylor theorem and hybrid algorithm to solve it.

Theorem 3.1: Let, $\phi: R^{2} \rightarrow R, \phi(m, n)=2 m-n-$ $\sqrt{4 \mathrm{~m}^{2}+\mathrm{n}^{2}}$ or
$\phi: \mathrm{R}^{3} \rightarrow \mathrm{R}, \phi(\mathrm{m}, \mathrm{n}, \varepsilon)=2 \mathrm{~m}-\mathrm{n}-\sqrt{4 m^{2}+\mathrm{n}^{2}+\varepsilon}$, where
$m \geq 0, n \geq 0$, then $\phi(m, n)=0 \dot{\Leftrightarrow} m n=0$, and $\phi(m, n, \varepsilon)=$ $0 \dot{\Leftrightarrow} \mathrm{mn}=\frac{\varepsilon}{4}, m \geq 0, n \geq 0$

Proof: $\phi(\mathrm{m}, \mathrm{n})=0 \dot{\Leftrightarrow} 2 \mathrm{~m}-\mathrm{n}-\sqrt{4 \mathrm{~m}^{2}+\mathrm{n}^{2}}=0$
$\dot{\Leftrightarrow} 2 \mathrm{~m}-\mathrm{n}=\sqrt{4 \mathrm{~m}^{2}+\mathrm{n}^{2}} \dot{\Leftrightarrow}(2 \mathrm{~m}-\mathrm{n})^{2}=4 \mathrm{~m}^{2}+\mathrm{n}^{2}$
$\dot{\Leftrightarrow} 4 \mathrm{~m}^{2}+\mathrm{n}^{2}-4 \mathrm{mn}=4 \mathrm{~m}^{2}+\mathrm{n}^{2} \dot{\Leftrightarrow}-4 \mathrm{mn}=0 \dot{\Leftrightarrow} \mathrm{mn}=0$.
Also
$\phi(\mathrm{m}, \mathrm{n}, \varepsilon)=0 \dot{\Leftrightarrow} 2 \mathrm{~m}-\mathrm{n}-\sqrt{4 m^{2}+\mathrm{n}^{2}+\varepsilon}=0$
$\dot{\Leftrightarrow} 2 \mathrm{~m}-\mathrm{n}=\sqrt{4 m^{2}+\mathrm{n}^{2}+\varepsilon} \dot{\Leftrightarrow}(2 \mathrm{~m}-\mathrm{n})^{2}=4 m^{2}+\mathrm{n}^{2}+\varepsilon$
$\dot{\Leftrightarrow} 4 m^{2}+\mathrm{n}^{2}-4 \mathrm{mn}=4 m^{2}+\mathrm{n}^{2}+\varepsilon \dot{\Leftrightarrow}-4 \mathrm{mn}=\varepsilon \dot{\Leftrightarrow} \mathrm{mn}$
$=\frac{\varepsilon}{4}, m \geq 0, n \geq 0$.
Using the proposed function $\phi(m, n, \varepsilon)=2 m-n-$ $\sqrt{m^{2}+n^{2}-\varepsilon}$ in problem (10), we obtain the following problem:
$\min _{\mathrm{x}} F_{1}(\mathrm{x}, \mathrm{y}, \mathrm{z})=a_{1} \mathrm{x}+b_{1} \mathrm{y}+c_{1} z$ s.t
$A_{1} \mathrm{x}+B_{1} \mathrm{y}+C_{1} \mathrm{z}-r_{1} \leq 0$,
$A_{2} \mathrm{x}+B_{2} \mathrm{y}+C_{2} \mathrm{z}-r_{2} \leq 0$,
$A_{3} \mathrm{x}+B_{3} \mathrm{y}+C_{3} \mathrm{z}-r_{3} \leq 0$,
$2 \mu_{\mathrm{i}}-\mathrm{g}_{\mathrm{i}}(\mathrm{x}, \mathrm{y})-\sqrt{4 \mu_{\mathrm{i}}^{2}+\mathrm{g}_{\mathrm{i}}^{2}(\mathrm{x}, \mathrm{y})-\varepsilon}=\frac{\varepsilon}{4}, \mathrm{i}=1,2, \ldots, \mathrm{l}$,
$\mu B_{3}=-b_{3}$,
$2 \beta_{\mathrm{i}}-\mathrm{h}_{\mathrm{i}}(\mathrm{x}, \mathrm{y})-\sqrt{4 \beta_{\mathrm{i}}^{2}+\mathrm{h}_{\mathrm{i}}^{2}(\mathrm{x}, \mathrm{y})-\varepsilon}=\frac{\varepsilon}{4}, \mathrm{i}=1,2, \ldots, \mathrm{l}$,
$\beta C_{2}=-c_{2}$,
$\mathrm{x}, \mathrm{y}, \mathrm{z}, \mu_{\mathrm{i}}, \beta_{\mathrm{i}} \geq 0$.
Which in the first constraint $m=\mu_{i} \geq 0, n=-g_{i}(x, y) \geq$ $0, \mathrm{~g}_{\mathrm{i}}(\mathrm{x}, \mathrm{y})=a^{i} \mathrm{x}+b^{i} \mathrm{y}+c^{i} \mathrm{z}$ and $a^{i}, b^{i}, c^{i}$ are i-th row of A , $B, C$ respectively and in the second constraint $m=\beta_{i} \geq 0, n=$ $-\mathrm{h}_{i}(x, y) \geq 0, \quad \mathrm{~h}_{\mathrm{i}}(\mathrm{x}, \mathrm{y})=a^{i} \mathrm{x}+b^{i} \mathrm{y}+c^{i} \mathrm{z}-\mathrm{r}$ and $a^{i}, b^{i}, c^{i}$ are i-th row of $\mathrm{A}, \mathrm{B}, \mathrm{C}$.
Let:
$G(x, y, \mu)=\left[\begin{array}{c}2 \mu_{1}-g_{1}(x, y)-\sqrt{\mu_{1}^{2}+g_{1}^{2}(x, y)-\varepsilon} \\ 2 \mu_{2}-g_{2}(x, y)-\sqrt{\mu_{2}^{2}+g_{2}^{2}(x, y)-\varepsilon} \\ \vdots \\ 2 \mu_{1}-g_{1}(x, y)-\sqrt{\mu_{1}^{2}+g_{1}^{2}(x, y)-\varepsilon}\end{array}\right]$
$H(x, y, \beta)=\left[\begin{array}{c}2 \beta_{1}-h_{1}(x, y)-\sqrt{\mu_{1}^{2}+h_{1}^{2}(x, y)-\varepsilon} \\ 2 \beta_{2}-h_{2}(x, y)-\sqrt{\mu_{2}^{2}+h_{2}^{2}(x, y)-\varepsilon} \\ \vdots \\ 2 \beta_{1}-h_{1}(x, y)-\sqrt{\mu_{1}^{2}+h_{1}^{2}(x, y)-\varepsilon}\end{array}\right]$
$H^{\prime}(x, y, \mu)=H(x, y, \mu)-\frac{\varepsilon}{4}, G^{\prime}(x, y, \mu)=G(x, y, \beta)-\frac{\varepsilon}{4}$
Problem (11) can be written as follows:
$\min _{\mathrm{x}} F_{1}(\mathrm{x}, \mathrm{y}, \mathrm{z})=a_{1} \mathrm{x}+b_{1} \mathrm{y}+c_{1} \mathrm{z}$
s.t
$A_{1} \mathrm{x}+B_{1} \mathrm{y}+C_{1} \mathrm{z}-r_{1} \leq 0$,
$A_{2} \mathrm{x}+B_{2} \mathrm{y}+C_{2} \mathrm{z}-r_{2} \leq 0$,
$A_{3} \mathrm{x}+B_{3} \mathrm{y}+C_{3} \mathrm{z}-r_{3} \leq 0$,
$G^{\prime}(\mathrm{t})=0$,
$H^{\prime}(\mathrm{t})=0$,
$\mu \mathrm{B}=-b_{3}$,
$\beta \mathrm{C}=-c_{2}$,
$x, y, z, \mu, \beta \geq 0$.
Where $t=(\mathrm{x}, \mathrm{y}, \mu) \in R^{k+2 l}$
Because problem (10) equal to (15), we use the following method for solving problem (15).

## The proposed algorithm based on Taylor method (TA)

Definition 4.1: The pair ( $\mathrm{X}, \mathrm{d}$ ) is a metric space which X is a non-empty set and d is a metric on X and:
$d \geq 0, \quad d(x, y)=0 \dot{\Leftrightarrow} x=y, \quad d(x, y)=d(y, x), \quad d(x, y) \leq$ $d(x, z)+d(z, y)$.

Definition 4.2: A sequence $\left\{x_{n}\right\}$ is said to Cauchy if for every $\varepsilon>0$ there is an N such that
$\forall_{m>r>N}\left|x_{m}-x_{r}\right|<\varepsilon$.
Theorem 4.1: All polynomials are continuous on real numbers. Additionally $x^{n}, \sqrt[n]{x}$ are continuous for every x ., when n is odd and for $\mathrm{x}>0$, when n is even.

Proof: The proof of this theorem has been proposed in 37.
Theorem 4.2: Suppose that two functions $g$ and $h$ are continuous at $\mathrm{x}=\mathrm{b}$. Then $g+h, g-h$ are continuous too at $\mathrm{x}=\mathrm{b}$.

Proof: The proof has been given by 37 .
Theorem 4.3: $\operatorname{Letlim}_{x \rightarrow a} g(x)=L \quad$ and f is continuous at L . Then,
$\operatorname{limf}_{x \rightarrow a}(g(x))=\mathrm{f}(\underset{x \rightarrow a}{\operatorname{limg}(\mathrm{x})})=\mathrm{f}(L)$

Proof: The proof has been given by 37 .
Corollary 4.1: Let g is continuous at a and f is continuous at $\mathrm{g}(\mathrm{a})$. Then, fog is continuous at a.

Proof: From above theorem, we have:
$\begin{aligned} \lim _{x \rightarrow a}(f o g)(x)= & \operatorname{limf}_{x \rightarrow a}(g(x))=\mathrm{f}(\operatorname{limg}(\mathrm{x})) \\ & =\mathrm{f}(g(a)) \\ & =(f o g)(a) . \text { since } g \text { is continuous at } a .\end{aligned}$
This finished the proof.
Theorem 4.4 (Taylor Theorem): Let f has $\mathrm{n}+1$ continuous derivatives on an open interval containing a. Then for each $x$ in the interval,
$f(x)=\left[\sum_{k=0}^{n} \frac{f^{k}(a)}{k!}(x-a)^{k}\right]+R_{n+1}(x)$
Where the error term $R_{n+1}(x)$, for some $c$ between a and $x$, satisfies
$R_{n+1}(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$
This form for the error $\mathrm{R}_{\mathrm{n}+1}(\mathrm{x})$, is called the Lagrange formula for the reminder.
The infinite Taylor series converge to $f$,
$f(x)=\left[\sum_{k=0}^{\infty} \frac{f^{k}(a)}{k!}(x-a)^{k}\right]$
If only iflim $\mathrm{n}_{\mathrm{n} \rightarrow \infty} \mathrm{R}_{\mathrm{n}+1}(\mathrm{x})=0$.
Proof: The proof of this theorem was given by 37 .
Using above Theorems and corollary, functions $G^{\prime}$ and $H^{\prime}$ in (15) are always continuous everywhere and it is possible to use Taylor Theorem for them in (15).

By applying the Taylor theorem to a feasible point such as $t^{k}$ for functions $G^{\prime}$ and $H^{\prime}$, and taking only two linear part of them in problem (15), the following linear functions are constructed: $G_{i}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla G_{\mathrm{i}}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)=0, \quad \mathrm{i}=1,2, \ldots \mathrm{l}$.
Let
$\mathrm{P}(\mathrm{t})=\left[\begin{array}{c}P_{1}(\mathrm{t}) \\ P_{2}(t) \\ \vdots \\ P_{l}(t)\end{array}\right]=\left[\begin{array}{c}G_{1}{ }_{1}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla G_{1}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right) \\ G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right) \\ \vdots \\ G_{l}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla G_{1}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)\end{array}\right]$
$\mathrm{H}_{\mathrm{i}}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla H_{\mathrm{i}}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)=0, \quad \mathrm{i}=1,2, \ldots \mathrm{l}$.
Let
$\mathrm{Q}(\mathrm{t})=\left[\begin{array}{c}Q_{1}(\mathrm{t}) \\ Q_{2}(t) \\ \vdots \\ Q_{l}(t)\end{array}\right]=\left[\begin{array}{c}H_{1}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla H_{1}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right) \\ H_{2}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right) \\ \vdots \\ H_{l}^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla H_{1}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)\end{array}\right]$
The obtained problem by using Taylor theorem is linear programming and can be solved using linear algorithm such as simplex method.

Initialization: The feasible point $t^{1}$ is created randomly, error $\varepsilon_{1}$ is given and it is supposed that $k=1, F(t)=F_{1}(\mathrm{x}, \mathrm{y}, \mathrm{z})=$ $a_{1} \mathrm{x}+b_{1} \mathrm{y}+c_{1} z, \varepsilon_{1}$ is a small and appropriate given error and finishing the algorithm depends on $\varepsilon_{1}$ such that it is finished whenever the difference between produced solutions by the algorithm in two consecutive iterations is less than $\varepsilon_{1}$.

Finding solution: Using Taylor theorem for $G^{\prime}$ and $H^{\prime}$ at $t^{k}$ and (20), (22), in problem (15) we obtain the following problem:
$\min _{\mathrm{t}} \mathrm{F}(t)$
$\quad \mathrm{s} . \mathrm{t}$
$A_{1} \mathrm{x}+B_{1} \mathrm{y}+C_{1} \mathrm{z}-r_{1} \leq 0$,
$A_{2} \mathrm{x}+B_{2} \mathrm{y}+C_{2} \mathrm{z}-r_{2} \leq 0$,
$A_{3} \mathrm{x}+B_{3} \mathrm{y}+C_{3} \mathrm{z}-r_{3} \leq 0$,
$\mathrm{P}(\mathrm{t})=0$,
$\mathrm{Q}(\mathrm{t})=0$,
$\mu \mathrm{B}=-b_{3}$,
$\beta \mathrm{C}=-c_{2}, \quad \mathrm{t}, \mu, \beta \geq 0$.

Making the present best solution: Because (23) is an approximation for (15) by Taylor theorem, therefore, the optimal solution for (23) is an approximation of the optimal solution for (15). Thus $t^{k+1}$ can be a good approximation of optimal solution problem (15). Therefore, we let $t^{*}=t^{k+1}$ and go to the next step.

Termination: If $\mathrm{d}\left(\mathrm{F}\left(\mathrm{t}^{\mathrm{k}+1}\right), \mathrm{F}\left(\mathrm{t}^{\mathrm{k}}\right)\right)<\varepsilon_{1}$ then the algorithm is finished andt* is the best solution by the proposed algorithm. Otherwise, we suppose $\mathrm{k}=\mathrm{k}+1$ and go back to the step 2. Which d is metric and,
$\mathrm{d}\left(\mathrm{F}\left(\mathrm{t}^{\mathrm{k}+1}\right), \mathrm{F}\left(\mathrm{t}^{\mathrm{k}}\right)\right)=\left(\sum_{i=1}^{n+2 m}\left(F\left(t_{i}^{k+1}\right)-F\left(t_{i}^{k}\right)\right)^{2}\right)^{\frac{1}{2}}$.
Following theorems show that the proposed algorithm is convergent.

Theorem 4.5: Every Cauchy sequence in real line and complex plan is convergent.

Proof: Proof of this theorem is given in [34].
Theorem 4.6: Sequence $\left\{F_{k}\right\}$ which was proposed in above algorithm is convergent to the optimal solution, so that the algorithm is convergent.

Proof: Let $\left(F_{l}\right)=\left(F\left(t^{l}\right)\right)=\left(F\left(t_{1}^{l}\right), F\left(t_{2}^{l}\right), \ldots, F\left(t_{n+2 m}^{l}\right)\right)=$ $\left(F_{1}^{(l)}, F_{2}^{(l)}, \ldots, F_{n+2 m}^{(l)}\right)$.
According to step 4
$d\left(F_{k+1}, F_{k}\right)=\mathrm{d}\left(\mathrm{F}\left(\mathrm{t}^{\mathrm{k}+1}\right), \mathrm{F}\left(\mathrm{t}^{\mathrm{k}}\right)\right)=\left(\sum_{i=1}^{n+2 m}\left(F\left(t_{i}^{k+1}\right)-\right.\right.$ $\left.\left.F\left(t_{i}^{k}\right)\right)^{2}\right)^{\frac{1}{2}}<\varepsilon_{1}$

The proposed algorithm has following steps:

Therefore $\left(\sum_{i=1}^{n+2 m}\left(F\left(t_{i}^{k+1}\right)-F\left(t_{i}^{k}\right)\right)^{2}\right)<\varepsilon_{1}{ }^{2}$. There is large number such as N which $\mathrm{k}+1>\mathrm{k}>\mathrm{N}$ and $\mathrm{j}=1,2, \ldots, 2 \mathrm{~m}+\mathrm{n}$ we have: $\left(F_{j}^{(k+1)}-F_{j}^{(k)}\right)^{2}<\varepsilon_{1}{ }^{2}$, therefore $\left|F_{j}^{(k+1)}-F_{j}^{(k)}\right|<\varepsilon_{1}$
Now let $m=k+1, r=k$ then we have
$\forall_{m>r>N}\left|F_{j}^{(m)}-F_{j}^{(r)}\right|<\varepsilon_{1}$.
This shows that for each fixed $\mathrm{j},(1 \leq \mathrm{j} \leq 2 \mathrm{~m}+\mathrm{n})$, the sequence $\left(\mathrm{F}_{\mathrm{j}}^{(1)}, \mathrm{F}_{\mathrm{j}}^{(2)}, \ldots\right)$ is Cauchy of real numbers, then it converges by theorem 4.5.

Say, $F_{j}^{(m)} \rightarrow F_{j}$ as $m \rightarrow \infty$. Using these $2 \mathrm{~m}+\mathrm{n}$ limits, we define $F=\left(F_{1}, F_{2}, \ldots, F_{2 m+n}\right)$. From (21) and $\mathrm{m}=\mathrm{k}+1, \mathrm{r}=\mathrm{k}, d\left(F_{m}, F_{r}\right)<$ $\varepsilon_{1}$

Now if $\mathrm{r} \rightarrow \infty, \mathrm{by}_{r} \rightarrow \mathrm{~F}$ we have $d\left(F_{m}, F\right) \leq \varepsilon_{1}$.
This shows that F is the limit of $\left(F_{m}\right)$ and the sequence is convergent by definition 3.3 therefore proof of theorem is finished.

Theorem 4.7: If sequence $\left\{f\left(\mathrm{t}_{\mathrm{k}}\right)\right\}$ is converge to $\mathrm{f}(\mathrm{t})$ and f be linear function then $\left\{t_{k}\right\}$ is converge to $t$.

Proof: Proof of this theorem is given in [34].
Theorem 4.8: Problems (15) and (23) are equal therefore they have same optimal solutions.

Proof: It is sufficient we prove that, $\left|G^{\prime}(t)-P(t)\right|<\varepsilon$ and $\left|H^{\prime}(t)-Q(t)\right|<\varepsilon$ for every arbitrary $\varepsilon>0$. According to the theorem 4.4 and (19), (20) we have:
$\mathrm{P}(\mathrm{t})=G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)$
$G^{\prime}(\mathrm{t})=G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)+\nabla^{2} G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right) \frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}+R_{n}(\mathrm{t})$.
$\left|G^{\prime}(t)-P(t)\right|=\left|\nabla^{2} G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right) \frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}+R_{n}(t)\right|$

$$
\leq\left|\nabla^{2} G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right) \frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}\right|+\left|R_{n}(t)\right|
$$

Now if $n \rightarrow \infty$, from (18) $\left|R_{n}(t)\right|<\frac{\varepsilon}{2}$ and let $\left|\nabla^{2} G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\right|<m$ that m is an arbitrary large number, this is possible because $\nabla^{2} G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)$ is a number.

If $k \rightarrow \infty$ because F is linear then by theorems 4.6 and 4.7 $\mathrm{t}^{\mathrm{k}} \rightarrow \mathrm{t}$ therefore $\left|\mathrm{t}^{\mathrm{k}}-\mathrm{t}\right|<\varepsilon_{2}$, say $\varepsilon_{2}=\sqrt{\frac{\varepsilon}{m}}$

$$
\begin{aligned}
\Rightarrow\left|G^{\prime}(t)-P(t)\right| & \leq\left|\nabla^{2} G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right) \frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}\right|+\left|R_{n}(t)\right| \\
& \leq\left|\nabla^{2} G^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\right|\left|\frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}\right|+\left|R_{n}(t)\right| \\
& \leq m \cdot \frac{\varepsilon}{2 m}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}
$$

Now we prove $\left|H^{\prime}(t)-Q(t)\right|<\varepsilon, \quad \mathrm{Q}(\mathrm{t}) \quad=H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+$ $\nabla H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)$
$H^{\prime}(\mathrm{t})=H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)+\nabla H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)+\nabla^{2} H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right) \frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}+R_{n}(t)$.
$\left|H^{\prime}(t)-Q(t)\right|=\left|\nabla^{2} H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right) \frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}+R_{n}(t)\right|$

$$
\leq\left|\nabla^{2} H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right) \frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}\right|+\left|R_{n}(t)\right|
$$

Now if $n \rightarrow \infty$, from (18) $\left|R_{n}(t)\right|<\frac{\varepsilon}{2}$ and let $\left|\nabla^{2} H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\right|<m$ that m is an arbitrary large number, this is possible because $\nabla^{2} H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)$ is a number.
If $k \rightarrow \infty$ because F is linear then by theorems 4.6 and 4.7 $\mathrm{t}^{\mathrm{k}} \rightarrow \mathrm{t}$ therefore $\left|\mathrm{t}^{\mathrm{k}}-\mathrm{t}\right|<\varepsilon_{2}$, say $\varepsilon_{2}=\sqrt{\frac{\varepsilon}{m}}$

$$
\begin{aligned}
\Rightarrow\left|H^{\prime}(t)-Q(t)\right| & \leq\left|\nabla^{2} H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right) \frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}\right|+\left|R_{n}(t)\right| \\
& \leq\left|\nabla^{2} H^{\prime}\left(\mathrm{t}^{\mathrm{k}}\right)\right|\left|\frac{\left(\mathrm{t}-\mathrm{t}^{\mathrm{k}}\right)^{2}}{2}\right|+\left|R_{n}(t)\right| \\
& \leq m \cdot \frac{\varepsilon}{2 m}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}
$$

This finished proof of theorem.

## Hybrid algorithm (HA)

We use a penalty function to convert problem (23) to an unconstraint problem.Consider problem (23); we append all constraints to the first level objective function with a penalty for each constraint.Then, we obtain the following penalized problem.

$$
\begin{align*}
\min _{\mathrm{t}} \mathrm{~F}(t)+\alpha_{1}(\beta \mathrm{C} & \left.+c_{2}\right)^{2}+\alpha_{2}\left(\mu B+b_{3}\right)^{2}+\alpha_{3}(\mathrm{P}(\mathrm{t}))^{2} \\
& +\alpha_{4}(\mathrm{Q}(\mathrm{t}))^{2} \\
& +\sum_{i} \alpha_{i}\left(A_{i} \mathrm{x}+B_{i} \mathrm{y}+C_{i} \mathrm{z}-r_{i}\right)^{2}(25) \tag{25}
\end{align*}
$$

Now we solve problem (23) using our line search method. The line search method is proposed as follows:
$x$ is a vector and $d$ is a direction, and $f$ is the smallest value from $x$ in the direction $d$. Our method searches along the directions $\quad\left(d_{1}, d_{2}, \ldots, d_{n-1}\right)$ where $d_{j}, j=1,2, \ldots, n-1$ is a
vector of zeros except at the $j$ th position which is 1 and $d_{n}=$ $\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$.
Clearly, all directions have a norm equal to 1 and they are linearly independent search directions. In fact, the proposed line search method uses the following directions as the search directions:
$d_{1}=(1,0, \ldots, 0), d_{2}=(0,1, \ldots, 0), \ldots, d_{n-1}=(0, \ldots, 1,0), d_{n}=$ $\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$

Therefore, along the search direction $d_{j}, j=1,2, \ldots, n-1$, the variable $x_{j}$ is changed but other variables will not be changed.The proposed line search method is explained for minimizing a function of different variables. Convergence of the presented algorithm will be proposed with the differentiable $f$.

Initial step: $\varepsilon>0$ is a small scalarwhich is used for finishing the algorithm, and let $d_{1}, d_{2}, \ldots, d_{n-1}$ be the coordinate directions and $d_{n}$ be a vector of $\frac{1}{\sqrt{n}}$. Choose an initial point $x_{1}$ let $x_{1}=y_{1} \cdot k=j=1$, and go to the next step.

Main step: Let $\mu_{j}$ which is optimal solution to $\operatorname{minimize}\left(y_{j}+\right.$ $\mu d_{j}$ ), and lety $y_{j+1}=y_{j}+\mu_{j} d_{j}$
If $j<n$ replace $j$ by $j+1$, and repeat step1. Otherwise, if $j=$ $n$, go to the next step.

Termination: Let $x_{k+1}=y_{n+1}$ if $\left\|x_{k+1}-x_{k}\right\|<\varepsilon$ then stop, otherwise, let $y_{1}=x_{k+1}$ and $j=1$, replace $k$ by $k+1$, and repeat step2.

We now propose a theorem which establishes the convergence of algorithms for solving a problem andtwo problems (15) and (25) have the same optimal solution.

## Theorem 5.1:

Consider the following problem:

$$
\begin{align*}
& \min _{x} f(x) \\
& \text { s.t } g_{i}(x) \leq 0, \mathrm{i}=1,2, \ldots, \mathrm{~m},  \tag{27}\\
& h_{j}(x)=0, \mathrm{j}=1,2, \ldots, 1
\end{align*}
$$

where $f, g_{1}, \ldots, g_{m}, h_{1}, \ldots, h_{l}$ are continuous functions on $R^{n}$ and $X$ is a nonempty set in $R^{n}$. Suppose that the problem has a feasible solution, and $\alpha$ is a continuous function as follows:

$$
\begin{equation*}
\alpha(\mathrm{x})=\sum_{i=1}^{m} \emptyset\left[g_{i}(x)\right]+\sum_{i=1}^{l} \emptyset\left[h_{i}(x)\right] \tag{28}
\end{equation*}
$$

where
$\emptyset(y)=0$ if $\mathrm{y} \leq 0, \quad \emptyset(y)>0$ if $y>0$.
$\phi(y)=0$ if $\mathrm{y}=0, \quad \emptyset(y)>0$ if $y \neq 0$.

Then,

$$
\begin{align*}
\inf \{f(x): g(x) \leq & 0, \quad h(x)=0, x \in X\} \\
& =\inf \{f(x)+\mu \alpha(x): x \in X\} \tag{31}
\end{align*}
$$

Where $\mu$ is a large positive constant $(\mu \rightarrow \infty)$.
Proof: This theorem has been proven by [29].

## Computational results

To illustrate both algorithms, we consider the following examples.

Example [38]: Consider the following linear tri-level programming problem:
$\min _{x} x-4 y+2 z$

$$
\begin{aligned}
& \text { s.t } \\
& -x-y \leq-3 \\
& -3 x+2 y-z \geq-10
\end{aligned}
$$

$\min _{\mathrm{y}} x+y-z$

$$
\begin{array}{r}
\text { s.t } \\
-2 \mathrm{x}+\mathrm{y}-2 \mathrm{z} \leq-1 \\
2 \mathrm{x}+\mathrm{y}+4 \mathrm{z} \leq 14
\end{array}
$$

$\min _{\mathrm{y}} x-2 y-2 z$

$$
\begin{aligned}
& \text { s.t } \\
& \quad 2 x-y-z \leq 2, \\
& x, y, z \geq 0 .
\end{aligned}
$$

Using KKT conditions, the following problem is obtained:
$\min _{x} x-4 y+2 z$
s.t
$-x-y \leq-3$,
$3 x-2 y+z \leq 10$,
$-2 x+y-2 z \leq-1$,
$2 \mathrm{x}+\mathrm{y}+4 \mathrm{z} \leq 14$,
$\beta_{1}(-2 \mathrm{x}+\mathrm{y}-2 \mathrm{z}+1)=0$,
$\beta_{2}(2 x+y+4 z-14)=0$,
$\beta_{1}+\beta_{2}=1$,
$2 \mathrm{x}-\mathrm{y}-\mathrm{z} \leq 2$,
$\mu(2 x-y-z-2)=0$,
$\mu(-1)=-2$,
$\mathrm{x}, \mathrm{y}, \mathrm{z}, \beta_{1}, \beta_{2}, \mu \geq 0$.
By the proposed function, the above problem becomes:
$\min _{x} x-4 y+2 z$
s. ${ }^{\mathrm{x}} \mathrm{t}$
$-x-y \leq-3$,
$3 x-2 y+z \leq 10$,
$2 \beta_{1}-(-2 \mathrm{x}+\mathrm{y}-2 \mathrm{z}+1)-\sqrt{\beta_{1}{ }^{2}+(-2 \mathrm{x}+\mathrm{y}-2 \mathrm{z}+1)^{2}+\square}=0$,
$2 \beta_{2}-(2 x+y+4 z-14)-\sqrt{\beta_{2}{ }^{2}+(2 x+y+4 z-14)^{2}+\square}=0$,

```
\(2 \mu-(2 x-y-z-2)-\sqrt{\mu^{2}+(2 x-y-z-2)^{2}+\square}=0\),
\(\beta_{1}+\beta_{2}=1\),
\(\mu(-1)=-2\),
    \(\mathrm{x}, \mathrm{y}, \mathrm{z}, \beta_{l}, \beta_{2}, \mu \geq 0\).
```

Using the Taylor theorem, we obtain a problem in the form of Eq. (23) and solve it using the proposed algorithm. Optimal solution is presented according to Table 1.

Table-1
Comparison of optimal solutions by Taylor algorithm Example 1

| Optimal <br> Solution | Best solution by our method <br> with different values of $\boldsymbol{\varepsilon}$ <br> $\boldsymbol{\varepsilon}=\mathbf{0 . 0 1}$ |  | Best solution <br> according to <br> reference38. |
| :--- | :---: | :---: | :---: |
| $\left(\square^{*}, \square^{*}, \square^{*}\right)$ | $(4.1,5.9,0)$ | $(4,6,0)$ | $(4,6,0)$ |
| $\square_{I}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ | -19.5 | -20 | -20 |
| $\square_{2}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ | 10 | 10 | 10 |
| $\square_{3}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ | -7.5 | -8 | -8 |

Example (solving by hybrid algorithm): Consider the following linear tri-level programming problem:

$$
\begin{aligned}
& \min _{x} x-4 y+2 z \\
& \quad \text { s.t } \\
& -x-y \leq-3 \\
& \quad-3 x+2 y-z \geq-10, \\
& \min _{y} x+y-z
\end{aligned}
$$

y

$$
\begin{array}{r}
\text { s.t } \\
-2 \mathrm{x}+\mathrm{y}-2 \mathrm{z} \leq-1, \\
2 \mathrm{x}+\mathrm{y}+4 \mathrm{z} \leq 14,
\end{array}
$$

$\min _{\mathrm{y}} x-2 y-2 z$

$$
\begin{aligned}
& \text { s.t } \\
& \quad 2 x-y-z \leq 2, \\
& x, y, z \geq 0 \text {. }
\end{aligned}
$$

Using hybrid algorithm the problem is solved. Optimal solution for this example by hybrid algorithm is presented according to Table 3.

Table-3
Comparison of optimal solutions by hybrid algorithm Example 1

| Optimal <br> Solution | Best solution by <br> hybrid algorithm | Best solution <br> according to <br> reference 38. |
| :---: | :---: | :---: |
| $\left(\square^{*}, \square^{*}, \square^{*}\right)$ | $(4.3,6.2,0.1)$ | $(4,6,0)$ |
| $\square_{1}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ | -20.3 | -20 |
| $\square_{2}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ | 10.4 | 10 |
| $\square_{3}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ | -8.3 | -8 |

Example 2 is solved by hybrid algorithm and computational results are proposed in table 4.

Table-4
Comparison of optimal solutions by hybrid algorithm
Example 2

| Optimal <br> Solution | Best solution by <br> hybrid algorithm | Best solution <br> according to <br> reference 38. |
| :--- | :---: | :---: |
| $\left(\square^{*}, \square^{*}, \square^{*}\right)$ | $(10.1,28.4,11.6)$ | $(10,28.33,11.66)$ |
| $\square_{1}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ | 147.16 | 146.66 |
| $\square_{2}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ | 176.93 | 176.6 |
| $\square_{3}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ | 345.33 | 343.3 |

## Conclusion and future work

In this paper, we used the KKT conditions to convert the problem into a single level problem. Then, using the proposed function, the problem was made simpler and converted to a smooth programming problem. The smoothed problem was been solved, utilizing the first proposed algorithm based on Taylor theorem and hybrid algorithm. Comparing with the results of previous methods, our algorithms have better numerical results and present better solutions. The bestsolutions produced by proposed algorithms are feasible unlike the previous best solutions by other researchers.

In the future works, the following should be researched: i. Examples in larger sizes can be supplied to illustrate the efficiency of the proposed algorithms. ii. Showing the efficiency of the proposed algorithms for solving other kinds of TLP such as quadratic and non-linear TLP.

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