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Short Communication

Formation of Black hole for Inhomogeneous Pressure and Density

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Abstract

We present here the models describing perfect fluid collapse which generalize the homogeneous dust collapse solution by including non-zero pressures and inhomogeneities. It is shown that a black hole or any lighter bodies like white dwarf or neutron star will be generated as end product of gravitational collapse, rather than a body having naked singularity. It is also shown that non-spacelike trajectories can't escape from the central singularity, formed in this process.

Keywords: Collapse, singularity, pressure, inhomogeneity and black hole.

Introduction

In recent years, dynamics of black hole has witnessed many theoretical as well as practical developments. Although, few exact static or stationary models of black holes are well studied[1-5]. But, the actual formation of black hole or any lighter bodies like white dwarf or neutron star within the framework of a dynamical gravitational collapse is an arena where our knowledge is limited and hence it provides a good scope to discuss the same.

Black hole or any lighter bodies born when a massive star exhausts its nuclear fuel and then the process of collapse underwent and continues under the influence of its own gravitational fields as far as the actual physical scenario is concerned. For showing spherically symmetric collapse, which satisfies the actual physical scenario, a model is given[6]. In this model, a dust cloud undergoes a gravitational collapse to form a black hole because of formation of event horizon well in advance to the epoch of the formation of the space-time singularity. In order to study collapsing models with more general collapsing situations and to understand the black hole formation in more realistic collapse scenario, it becomes essential to study pressures and inhomogeneities in these models. It also helps to put one step forward towards the cosmic censorship hypothesis[7], which states that any physically realistic gravitational collapse must result into the development of a black hole. However, this hypothesis remains as a major unresolved problem in today's black hole physics.

Here, we study a specific class of collapse models which generalizes the Oppenheimer-Snyder dust collapse model by including pressures and inhomogeneities. The fluid content of the body in this class is in the form of a perfect fluid with an equation of state of the form $p = k\rho$. Although the case of a general inhomogeneous dust collapse with $k = 0$ has already been solved[8]. But, the fate of such collapsing models has not been discussed. Also, in recent present, an inhomogeneous charged black hole is discussed[9]. For the present study, we assume specific mass function, which is separable in the functions of physical radius of the body and the time coordinate. Similar kind of mass function is taken in[10]. But, our assumption facilitates us to discuss the final fate of collapse for larger value of mass. This fact also provides insights into the actual process of formation of a black hole or any lighter bodies. Also, the class of models described here proves essential to resolve the issue of cosmic censorship. Here, we examine the solutions of Einstein equations for spherically symmetric perfect fluid to discuss explicitly how a body with inhomogeneous density should behave in the later stages of collapse near the singularity so that the fate of collapse would always be a black hole or any lighter bodies.

Space-time: The spherically symmetric collapsing body can be described by the space-time geometry given by the metric in the comoving coordinates (r,θ,ϕ,t) as

$$ds^2 = -e^{2\upsilon(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)\left[d\theta^2 + \sin^2\theta d\phi^2\right] \quad (1)$$

Solution of field Equations

For solving the Einstein equations for the metric (1) in the units $8\pi G = c = 1$, we consider the energy-momentum tensor of the type I given as[11]

$$T_1^1 = p_r(t,r), T_2^2 = T_3^3 = p_\theta(t,r), T_4^4 = -\rho(t,r), \quad (2)$$

where ρ, p_r and p_θ are the energy density, radial and tangential pressures respectively of the body. We take the matter field to satisfy the weak energy condition i.e. $\rho(t,r)$ measured by any local observer is non-negative. So, for any time-like vector V^i, we must have

$$T_{ik}V^iV^k \ge 0 \quad (3)$$

which reduces to

$$\rho \ge 0, \rho + p_r \ge 0, \rho + p_\theta \ge 0 \quad (4)$$

Now, the Einstein equations are solved for metric (1) and energy-momentum tensor (2) as

$$\rho = \frac{M'}{R^2 R'} \quad , \quad p_r = -\frac{\dot{M}}{R^2 \dot{R}} \tag{5}$$

$$\upsilon' = \frac{2(p_\theta - p_r)}{\rho + p_r}\frac{R'}{R} - \frac{p_r'}{\rho + p_r} \tag{6}$$

$$-2\dot{R}' + R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} = 0 \tag{7}$$

and $$G - H = 1 - \frac{M}{R}, \tag{8}$$

where $$G(t,r) = e^{-2\psi}R'^2 \quad , \quad H(t,r) = e^{-2\upsilon}\dot{R}^2 \tag{9}$$

Here, the arbitrary function $M = M(t,r)$ has an interpretation of the mass function for the body, which provides the total mass in a shell of comoving radius r on any space-like slice $t =$ constant. Using the energy conditions, we have $M \geq 0$. In order to preserve the regularity at the initial epoch $t = t_i$, we should have $M(t_i, 0) = 0$ i.e. the mass function should vanish at the center of the body. As we are considering collapse scenario, we have $\dot{R} < 0$ i.e. the physical radius R of the cloud decreases continuously till it reaches at $R = 0$. It can easily be seen from equation (5) that a density singularity exist in the space-time at $R = 0$ and $R' = 0$, where singularity at $R' = 0$ is a weak singularity due to shell-crossings and it can be possibly removed from the space-time[12]. So, we consider here only the shell-focusing singularity at $R = 0$ as appropriate physical singularity. Now, by incorporating the perfect fluid form of matter given as

$$p_r(t,r) = p_\theta(t,r) = k\rho(t,r) \quad ; \quad k < 1 \text{ is a constant,} \tag{10}$$

equations (5) and (6) reduces to

$$\rho = \frac{M'}{R^2 R'} = -\frac{1}{k}\frac{\dot{M}}{R^2 \dot{R}} \tag{11}$$

and $$\upsilon' = -\frac{k}{k+1}\left[\log_e(\rho)\right]' \tag{12}$$

By using scaling independence, we can write $R(t_i, r) = r$ at the initial epoch $t = t_i$, from where the collapse commences. We define the time $t = t_s(r)$ corresponds to the formation of singularity at $R = 0$.

Now, we assume that the mass function can be written as a product of functions of R and t as

$$M(t,r) = M(R,t) = \frac{R^3 \mu(t)}{\xi(R)} \tag{13}$$

where $\xi(R)$ is a positive function of R and $\mu(t)$ will be determined by the field equations. Also, $\mu(t)$ is assumed as a differential function of t for $t < t_{s_0}$, where $t = t_{s_0}$ is the epoch for the occurrence of the central singularity. We choose this mass function as it is capable of introducing pressures and inhomogeneities, which allows us to construct collapse models which are more general and end up in a black hole or any lighter bodies as we shall see.

Also, we need smooth initial data i.e. at the initial epoch $t = t_i$, density and pressures must be smooth or analytic functions. So, we assume function $\xi(R)$ as

$$\xi(R) = 1 + \xi_2 R^2 + ... \tag{14}$$

Using above form of $\xi(R)$, one can easily justify the existence of $\xi(R)$ in denominator of $M(R,t)$ as it increases the mass and the density with decreasing R. Also, by using equation (14) in equation (13), equation (11) provides

$$\rho = \rho(t,r) = \rho(R,t) = \left[3\xi(R) - R\xi(R)_{,R}\right]\frac{\mu(t)}{[\xi(R)]^2} = \frac{3\eta(R)\mu(t)}{[\xi(R)]^2} \tag{15}$$

where the function $\eta(R)$ is given by

$$\eta(R) = 1 + \frac{1}{3}\xi_2 R^2 + ... \tag{16}$$

At the initial epoch $t = t_i$, using $\mu(t) = \mu(t_i)$ and $R(t_i, r) = r$, we have initial density

$$\rho(t_i, r) = \rho_0(r) = \frac{3\mu(t_i)\left(1 + \frac{1}{3}\xi_2 r^2 + ...\right)}{\left(1 + \xi_2 r^2 + ...\right)^2} \tag{17}$$

Here, we see that the gradients of the density and pressures of the body vanish at the center $r = 0$ at the initial epoch $t = t_i$ as required by the smoothness. Also, for diverging density at singularity, we must have

$$\lim_{t \to t_{s_0}} \mu(t) \to \infty \tag{18}$$

Using equation (5), (13) and (15), the perfect fluid condition can be written as

$$3(k+1)\mu(t)\eta(R) + \frac{R}{\dot{R}}\xi(R)\dot{\mu}(t) \tag{19}$$

By solving above equation, the mass function can be determined completely.

By combining equations (12) and (15), we get

$$\upsilon = -\frac{k}{k+1}\log\left[\frac{C_1\eta(R)}{\{\xi(R)\}^2}\right], \tag{20}$$

where C_1 is a constant of integration. Using equation (20), it is obvious to state that $\upsilon = \upsilon(R)$ i.e. the metric function υ is a function of the physical radius R only. Now, by substituting the value of $H(t,r)$ in equation (7), we get

$$R'\dot{G} - 2\dot{R}\upsilon' G = 0 \tag{21}$$

By solving above equation, the function $G(R)$ is given as

$$G(R) = \left[\eta(R)\right]^{-2k/(k+1)}\left[\xi(R)\right]^{4k/(k+1)} \tag{22}$$

Above forms of ρ, υ and G completely solve the Einstein equations after obtaining the value of $\mu(t)$.

By substituting values of G, H, M and υ in equation (8), we get

$$\dot{R} = -C_2\left[\eta(R)\right]^{-k/(k+1)}\left[\xi(R)\right]^{2k/(k+1)}\sqrt{\left[\eta(R)\right]^{-2k/(k+1)}\left[\xi(R)\right]^{4k/(k+1)} - 1 + \frac{R^2\mu(t)}{\xi(R)}} \tag{23}$$

where $C_2 = C_1^{-k/(k+1)}$ is a constant. Here, we choose negative sign for $\dot{R}$ so that it denotes the collapse scenario $\dot{R} < 0$. By substituting the values of $\eta(R)$ and $\xi(R)$ in equation (23) and by ignoring the higher order terms to get the solution close to the singularity, we get

$$\dot{R} = -C_2 R\left(1 + \frac{5}{3}\frac{k}{k+1}\xi_2 R^2\right)\sqrt{\left(1 - \xi_2 R^2\right)\mu(t) + \frac{10}{3}\frac{k}{k+1}\xi_2} \tag{24}$$

Now, we solve equation (19) close to the space-time singularity at $R = 0$. For this purpose, we use equation (24) and approximations $\eta(R) \to 1, \xi(R) \to 1$. It reduces equation (19) in the form

$$3(k+1)\mu(t) - \frac{1}{C_2\sqrt{\frac{10}{3}\frac{k}{k+1}\xi_2 + \mu(t)}}\dot{\mu}(t) = 0 \tag{25}$$

Solving the above equation with the boundary condition defined in equation (18), we get

$$\mu(t) = -\alpha^2 + \left[\alpha + \frac{2\alpha}{e^{-3C_2(k+1)\alpha(t_{s_0} - t)}}\right]^2 \tag{26}$$

where $$\alpha = \sqrt{\frac{10}{3}\frac{k}{k+1}\xi_2} \tag{27}$$

Form of $\mu(t)$ given in equation (26) is solution of Einstein equations in the vicinity of the singularity with respect to the given forms of ρ, υ and G. Also, for above form, we have $\mu(t) > 0$. By solving equation (24) in vicinity of singularity and by ignoring the higher order terms, we get

$$R(t,r) = g(r)e^{-P(t)} \tag{28}$$

where $g(r)$ is an arbitrary function of r. For avoiding any shell crossing singularity, we assume $g(r)$ to be an increasing function of r and $g(0) = 0$. In above equation (28), $P(t)$ is given as

$$P(t) = C_2\int\sqrt{\mu(t) + \frac{10}{3}\frac{k}{k+1}\xi_2}\;dt \tag{29}$$

Now, we have to decide the space-time singularity occurring at $R = 0$ is covered by an event horizon or it is naked. To decide this, we have to think over any future directed families of null geodesics which if terminate at the singularity in the past and go out to external observer in the future, then the singularity is naked, otherwise it is covered by an event horizon.

Results and Discussion

According to the form of R given in equation (28) as well as $\dot{R} < 0$, the singularity happens at the singular epoch $t = t_s$, where the physical radius for all the shells with different values of variable r becomes zero. In other words, as $t \to t_{s_0}$, all the shells collapse simultaneously to the singularity i.e. as $t \to t_{s_0}$, all the shells collapse simultaneously to the singularity at $R = 0$. This process generates a necessary covered central singularity at $R = 0, r = 0$, as there does not exist any outgoing future directed non space-like geodesics coming out from the same. Because, if these geodesics do exist in form of $t = t(r)$ in $t - r$ plane and comes out from $t = t_s, r = 0$, then the time coordinate must increase along these paths, which is not possible as there is complete collapse at epoch $t = t_s$ and beyond that no space-time exist.

Conclusion

Here, we find a singularity covered by an event horizon which may appear as a black hole or any lighter bodies as per the availability of mass at the epoch $t = t_s$, where the collapse ends.

Here, we generalized the homogeneous dust collapse model by including non-zero inhomogeneous pressures and density, which is more realistic physical scenario.

Although, here we have shown existence of black hole or any lighter bodies with special choices of mass function and velocity profile, but similar results can also be obtained by generalizing the choices of mass function and velocity profile up to some extent. Here, it is important to state that the covered singularity exist within certain limits over inhomogeneities[13]. Beyond these critical limits of inhomogeneities, the collapse could end in a naked singularity.

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