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The efficiency of spectral-element and finite-element methods in acoustic wave propagation

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Abstract

One of the great technique for surveying of the Earth's subsurface is to simulate seismic wave propagation using numerical modeling. Various numerical approaches are available for simulation of wave propagation in different media, including finite-difference method (FDM), discontinuous Galerkin method (DGM), finite-element method (FEM), finite volume method (FVM), and spectral-element method (SEM). Among different simulation approaches, FEM is a popular method in order to modelling of wave propagation because of flexibility and efficiency for simulation in complex geometries and inhomogeneous media. Standard FEM is an implicit method that means a linear system is required to be solved. Accordingly, it is a slower method that FDM as a result it limited the applicability to seismology. Solving such algorithms on parallel computers with distributed memory complicates matters further. In order to avoid this undesired problem, the spectral-element numerical approach is introduced for simulation of wave propagation. The formulations and equations of SEM is almost as same as that FEM with a tiny differences which makes it more suitable and optimal than finite-element method in the time-domain modelling. In fact, SEM is almost a new numerical technique for simulation of wave propagation. The purpose of this study is proposing the differences between the spectral-element method and finite-element method for simulating seismic wave propagation in different angle with straightforward formulation. The accuracy of the methods are shown by comparing the finite-element and spectral-element solutions with analytical solutions of the two-dimensional (2D) model. Numerical modeling examples show the great performance of the spectral-element scheme over finite-element method.

Keywords: Finite-element method, Numerical modelling, spectral-element method, wave propagation.

Introduction

Numerical modelling of seismic wave propagation has an indispensable function at almost every part of seismic exploration from survey design methods to inversion algorithms and imaging[1]. Full waveform inversion[2], 3D wave simulation[3], and reverse time migration[4], are examples of numerical modelling. The finite-element and the finite difference methods are most common numerical techniques in seismology[5]. The finite-difference method is the important and commonplace method for simulation of seismic waves which it is planned according to estimation that allow replacing differential equations by finite-difference equations[6].

The finite-element method is a strong and adaptable tool for applying to the wave propagation problem. The general formulation and the flexibility of media parameters and boundary conditions have made FEM a great technique to form general-purpose computer programs in order to solve a wide range of problems[7]. However, despite the flexibility and high accuracy of the standard FEM, the method is rarely used for numerical modelling of seismic waves because it requires enormous processing memory and too much calculations[8].

These limitations, make the wave propagation problem computationally and mathematically challengeable in FEM. The spectral-element methods (SEM) has been successfully used for simulation of the wave equation, addressing the limitations of FEM and providing fast numerical method than the FEM by using the weak formulation of motion equations[9]. The method includes the advantageous of FEM with the efficiency of a spectral method which it has been used for simulation of wave propagation in global and regional seismology that obtaining results remarkably are closed to the real data[10]. On the other hand, SEM also called a higher-order finite-element method (hp-FEM) that high-degree Lagrange interpolates is applied to discretize the wave field on the elements and Gauss-Lobatto-Legendre integration is utilized to solve integration over an element[11].

The main intention of this study is to examine the numerical responses of the acoustics wave equation solving according to FEM and SEM algorithms and discuss about accuracy, computational performance and other factors of different simulation for a 2D problem.

Theory: The acoustics wave equation describe the propagation of seismic waves in subsurface as following[12],

$$\frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p = S \quad (1)$$

where p is the acoustic pressure, c expresses the propagation velocity, S is external forces (source) and t is time. The variational or weak form of the wave equation will obtain as follows[5]:

$$\iint_{\Omega} \ddot{p}\, \delta p\, d\Omega + \iint_{\Omega} c^2 (\nabla p . \nabla \delta p)\, d\Omega - \iint_{\Gamma} c^2 (\nabla p . n)\, \delta p\, d\Gamma = \iint_{\Omega} S\, \delta p\, d\Omega \quad (2)$$

Where Ω expresses the region domain, n is unit outward normal, Γ presents the region boundary and the variation operator is given by δ. In SEM and FEM, an approximation of the unknown solution within an element is estimated according to the interpolation (shape) functions ψ_i ($i = 1,...,n$) that only rely on space. We represent this approximation by[13]:

$$p(x,t) = \sum_{i=1}^{M+1} \psi_i(x)\, p_i(t) \quad (3)$$

Where p_i expresses the nodal values of the model response at node i of the element, and ψ_i is the interpolation functions for node i. The polynomial degree, M, is utilized to show functions on each element, therefore $M + 1$ is the number of nodes for the element in each direction. Practically, if the value of M is too small, less than generally 5, a SEM modeling almost shows the same response and accuracy of a standard FEM that apply to wave propagation problems. On the other hand, when the polynomial degree is very large, more than 10, the method is spatially very exact, but the computational cost becomes excessive. Optimal value of polynomial degrees, M, is usually between 5 and 10 for spectral-element method usage in wave propagation problems[10]. The polynomial degree in the standard FEM is M =1. However, the variation of p can be expressed as matrix form in the following format:

$$\delta p = \psi\, \delta\, p \quad (4)$$

Following spatial discretization, regardless of boundary conditions, the displacement formulation of the acoustic wave equation 2 becomes an algebra-differential equation which can express in general form as following[14]:

$$\boldsymbol{M} . \ddot{p}(t) + \boldsymbol{K} . p(t) = \boldsymbol{S}(t) \quad (5)$$

and $\boldsymbol{M}$ expresses the mass matrix and $\boldsymbol{K}$ presents the stiffness matrix and that they can be written as:

$$\boldsymbol{M} = \iint_{\Omega_i} \boldsymbol{N}^T \boldsymbol{N}\, d\Omega \quad (6)$$

$$\boldsymbol{K} = \iint_{\Omega_i} c^2 \left(\frac{\partial \boldsymbol{N}^T}{\partial x} \frac{\partial \boldsymbol{N}}{\partial x} + \frac{\partial \boldsymbol{N}^T}{\partial z} \frac{\partial \boldsymbol{N}}{\partial z} \right) d\Omega \quad (7)$$

The presented formulations are as matrix format which streamlines the performance of them in programming environments, especially in MATLAB environment. We can solve the ordinary differential equations of equation 5 in the frequency-domainor in the time-domain based upon the specific application and the available computational resources[5]. Finite-difference approximations are suitable for modelling of wave propagation in the time-domain mostly by using the displacement of the time derivatives into the space-discrete equations[7]. Field $\ddot{p}$ also can be obtained by applying the second order of time derivative of the discrete displacement by using central difference approximations that explicitly obtain by:

$$\ddot{p}(t) = \boldsymbol{M}^{-1} . [\, \boldsymbol{S}(t) - \boldsymbol{K} . p(t)\,] \quad (8)$$

The result of this approximation is an explicit time-stepping scheme that allows us to compute the displacement at time $t + dt$ from the displacement at times t and $t - dt$. Where dt express the time stepping interval. Equation 8 can solved as linear simultaneous equations that we have used computer program, MATLAB, for solving it.

The finite-element and spectral-element

Standard finite-element method is similar in all given relationships to high-order finite-element method (spectral-element). Interpolating points and numerical integration method are two main differences between finite-element and spectral-element. Nodal points with identical space are used in the standard FEM. In fact, if nodal points with evenly-spaced are used, the interpolation accuracy will decline. The reason of this incident is Runge's phenomenon. Runge's phenomenon occur close the edges of the interpolation interval when identical space points are applied[5]. Figure-1a shows the Rung's phenomenon in interpolation of one-dimensional Gaussian function with equidistant nodal points. This undesirable effect can be removed by using Gauss-Lobatto-Legendre (GLL) Points for polynomial interpolation Figure-1b. As you can see, precision of interpolation has increased significantly with GLL nodal points.

Another difference between the methods is the numerical integration, Gaussian points are used as collocation points and integration weights in standard finite-element method, while GLL points are used in the spectral-element as the points of integration. Gaussian quadrature is used as numerical integration in both methods. The combination of discretization and applying the GLL quadrature method in order to achieve accurate estimation of the integrals of equation 6 leads to a diagonal mass matrix which considerably simplifies the algorithm in SEM and decreases drastically computational cost[15].

In order to better understanding the differences between FEM and SEM, a simple model, 50m×50m, with velocity of 3,000 m/s has considered. The maximum frequency is 60Hz.

Therefore, according to the wavelength equation ($\lambda = \frac{v}{f} = \frac{3000}{60}$), the smallest wavelength will be 50m.

Figures-2 and 3 show the patterns of mass and stiffness matrixes for two methods. As you can see, the mass matrix (Figure-2b) for the spectral element method is a single diagonal matrix, as a result, in the space discretization, the equations for each node is independent of other nodes thus there is no need to calculate the inverse of this matrix in equation 8. Therefore, it allows an efficient parallel implementation in programing environment especially in Matlab. In addition, the stiffness matrix for an element (Figure-2a) is completely filled. Therefore, the accuracy of the response in the spectrum method will be more than FEM.

Discretization of SEM wave field at the elements is done by using high-degree Lagrange interpolates, and integration of each element is resolved by using GLL integration rule. It causes lowest numerical grid dispersion and anisotropy. Diagonality of mass matrix is the most pivotal feature of the SEM, which because of this important feature, we can use a straight forward time integration scheme without having to invert a linear system (equation 8) that drastically reduces the computational cost and simplifies the implementation. Furthermore, it allows an efficient parallel implementation in programing environment especially in Matlab environment.

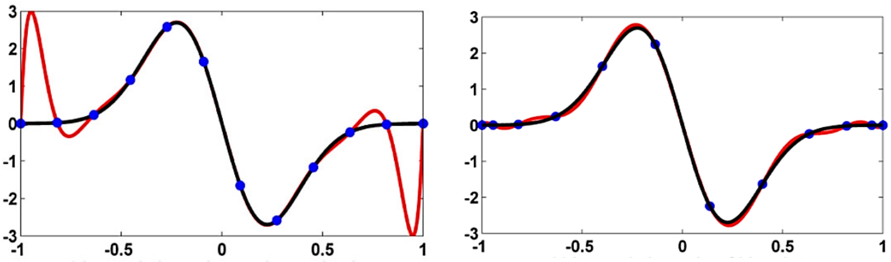

Figure-1: Runge's effect in FEM. Runge's function is Gaussian function, f (x) = e^{-10x^2} with a dominant frequency of 15 MHz (black curve) and the interpolant curves are red. The equidistant points in the interpolation of Runge's function (left) causes a noticeable overshooting of the interpolantnear at the edges of the interpolation interval [−1, 1]. By using the GLL points as interpolation points (right) this undesirable effect is removed.

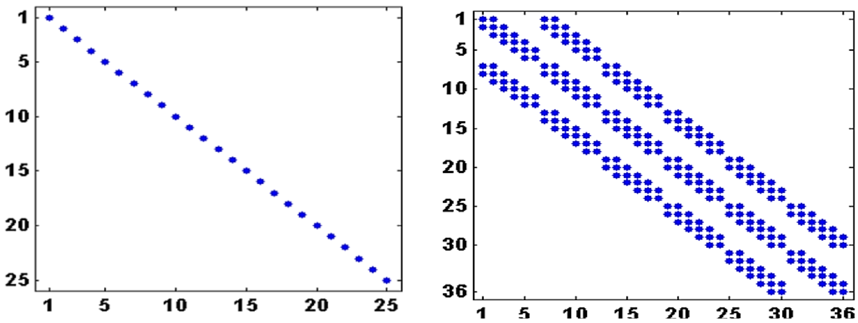

Figure-2: Mass matrices scheme. a: FEM, b: SEM.

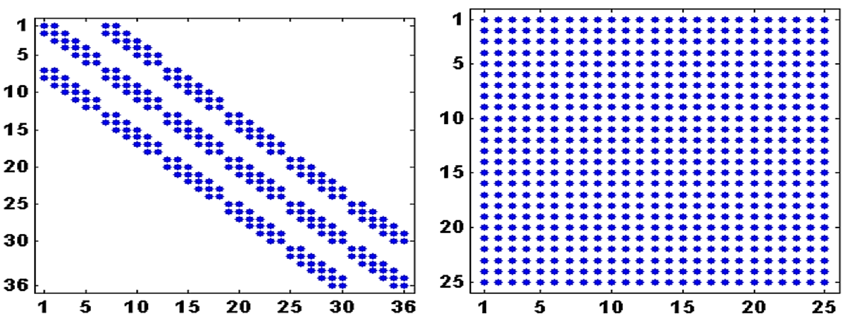

Figure-3: Stiffness Matrices scheme. a:FEM, b:SEM.

Numerical example

In order to investigate the accuracy and computational costs of FEM and SEM, we have considered a 2D homogeneous model, 1500m×1500m,withvelocity of 3,000m/s. Ricker wavelet with a dominant frequency of 30Hzis used as a source of seismic energy. Analytical estimates are calculated by using the software package "Gar6more2D".

We show numerical response for a typical receiver located at 600 meters far away from the source (Figure-4). Figure-4 compares the exact solution with 6th order spectral-element method (SEM 6), 8th order spectral-element method (SEM 8), and FEM. The results of the seismic analysis corresponding to the modelling characteristics of 6th order and 8th order spectral-element and the finite-element methods in Table-1.

As can be seen, the finite-element method in spite of using smaller elements has larger error. The 8th order SEM with 4.5 nodes, at the minimum wavelength, has very reasonable accuracy. SEM 6 has some numerical dispersion that magnification of the final part of answer is clearly visible.

We have written all algorithms in Matlab environment. According to Table-1, computational costs of FEM in Matlab is more than 20 times that of the SEM.

Evidently the most advantageous feature of the SEM is the diagonality of the mass matrix[15]. In addition, even though the size of elements in FEM are much smaller than that of SEM, but it has much lower accuracy. Since in the application of seismic oil exploration, information related to environmental features such as speed and density are also available at 25 or 50 meters; so the spectral-element method will be optimum. Since the dimensions of environment in applications of seismic oil exploration is bigger than 6km×20km, the problem has very high degree of freedom and requires very high memory and computational costs and according to the results reported in the Table-1, using the finite-element method in the desired applications will be practically impossible and more research on this technique does not recommend.

It should be noted that using SEM in applications other than oil exploration including issues of mechanical engineering, structural dynamics and vibration compared with finite-element method should be reviewed, because in these applications the dimensions are very small and the frequency content of energy sources is very high and the properties of the environment change at very lower intervals. In this case, in the spectral-element method should also use much smaller elements which will increase computational cost.

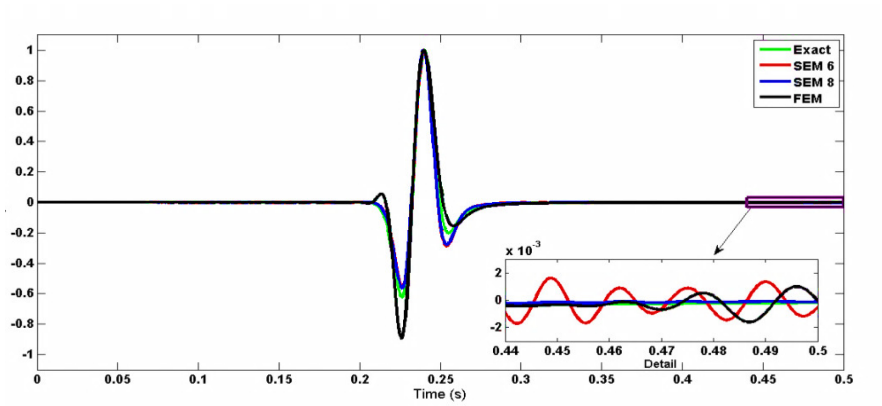

Figure-4: Comparison of numerical dispersion in SEM and FEM simulation of seismic wave propagation in 2D homogeneous medium. Responses are at 600*m* of the source. Numerical dispersion becomes insignificant, when the polynomial degree and grid points per wavelength are 8 and 4.5 respectively in SEM8 (Blue line). A Ricker wavelet with a dominant frequency of 30 MHz is used as a source. Black line shows numerical dispersion effect in SEM with the polynomial degree 6 and red line show FEM responses. Exact solution is shown by green line.

Table-1: Comparison of different factors in a simulation seismic waves by using SEM and FEM. SEM 6 is SEM with the polynomial degree 6 and SEM 8 represents SEM with the polynomial degree 8.

Method	Computation Time (s)	Degrees of freedom	The number of nodes in the smallest wavelength	Element length (m)	The smallest wavelength (m)	The standard error (%)
FEM	795	63001	6	5	30	10.93
SEM 6	14	22801	3.5	15	30	6.81
SEM 8	35	40401	4.5	20	30	2.72

Conclusion

We study numerical modelling of wave propagation in seismic based on spectral-element and finite-element methods and discuss about accuracy and computational cost of them for a 2D model. The properties of the spectral-element and the finite-element formulations are presented, and then the relations and differences between the two formulations are established. Numerical modelling examples demonstrate the performance of the spectral-element scheme. The results show that there is an optimum time and accuracy for SEM compared to FEM. In the application of seismic exploration, the finite-element method is undesirable and its usage in time-domain is practically impossible. The results show that SEM with the polynomial degree 8 of 4.5 grid points per minimum wavelength has very good accuracy and there is an acceptable and great accordance between the results of exact response and high order spectral-element, SEM 8.

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